THE ETA INVARIANT ON TWO-STEP NILMANIFOLDS

RUTH GORNET AND KEN RICHARDSON

ABSTRACT. The eta invariant appears regularly in index theorems but is known to be computable only in certain examples of locally symmetric spaces of compact type. In this work, we derive some general formulas useful for calculating the eta invariant on closed manifolds. Specifically, we study the eta invariant on nilmanifolds by decomposing the spin^c Dirac operator using Kirillov theory. In particular, for general Heisenberg three-manifolds, the spectrum of the spin^c Dirac operator and the eta invariant are computed in terms of the metric, lattice, and spin structure data. There are continuous families of geometrically, spectrally different Heisenberg three-manifolds whose Dirac operators have constant eta invariant. We also show that the Dirac spectrum is symmetric about zero in Heisenberg manifolds of dimension $4m+1$ ($m \in \mathbb{Z}_{>0}$); thus, the eta invariant is automatically zero in these dimensions. In the appendix, some needed results of L. Richardson and C. C. Moore are extended from spaces of functions to spaces of spinors.

1. Introduction

The eta invariant was introduced in the famous paper of M. F. Atiyah, V. K. Patodi, and I. M. Singer (see [3]), in order to produce an index theorem for manifolds with boundary. The eta invariant of a linear self-adjoint operator is roughly the difference between the number of positive eigenvalues and the number of negative eigenvalues, which of course is undefined when these numbers are both infinite. However, this quantity may be regularized to make it well-defined for classical pseudodifferential operators, using methods similar to the zeta-function regularization of the determinant of the Laplacian and methods used by physicists to regularize divergent integrals. The eta function is analogous to Dirichlet L-functions in the same way that the zeta function of elliptic operators is analogous to the Riemann zeta function.

Let $D : C^\infty(E) \to C^\infty(E)$ be an essentially self-adjoint elliptic classical pseudodifferential operator of order d on sections of a vector bundle $E \to M$, where M is a closed (compact, without boundary) Riemannian manifold of dimension n. Let $\{\lambda\}$ be the collection of eigenvalues with multiplicity. The eta function is defined as

$$\eta(s) = \sum_{\lambda \neq 0} \operatorname{sgn}(\lambda) |\lambda|^{-s}.$$

This reduces to the zeta function if D has only nonnegative eigenvalues. The eta function is holomorphic in s for large $\operatorname{Re}(s)$ and can be analytically continued to a meromorphic function using heat kernel techniques. It is true but not obvious that $\eta(s)$ is regular at $s = 0$, and $\eta(0)$ is always real; the eta invariant is defined as $\eta(0)$. See [3], [4], [16] for general information about the eta invariant.

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The eta function and its generalizations have been studied and utilized in index theorems for noncompact manifolds and for families of operators and in gluing formulas. The sign of the eta invariant of the boundary signature operator on a 4-manifold with boundary has important geometric content; in the case of a ball, it determines whether the conformal class of the boundary metric contains a metric induced from a self-dual Einstein metric on the interior (see [21]). In physics, the eta invariant of the spin and spin^c Dirac operators has practical importance, for example in the regularization of Feynman path integrals (see [30]). Recently, in the work of J. Brüning, F. W. Kamber, and K. Richardson, the eta invariant is utilized in a new equivariant index formula for G-manifolds and an index formula for Riemannian foliations (see [8], [9], [10]).

It is very difficult to calculate the eta invariant for a given operator such as a Dirac operator on a Riemannian manifold; much work has been done to calculate this invariant for space forms, lens spaces and flat tori (see, for example, [13], [15], [7]). More recently, S. Goette has calculated formulas for the eta invariant and equivariant eta invariants on homogeneous spaces of the form $G\diagup H$ with G compact (see [17]). The first work on computing eigenvalues of Dirac operators on homogeneous spaces corresponding to noncompact Lie groups has been done by B. Ammann and C. Bär (see [1], [5]), where the eigenvalues of the spin^c Dirac operator on certain (rectangular) Heisenberg manifolds were computed explicitly.

A Riemannian nilmanifold is a closed manifold of the form $(\Gamma\diagdown G, g)$ where G is a simply connected nilpotent Lie group, Γ is a cocompact (i.e., $\Gamma\diagdown G$ is compact) discrete subgroup of G, and g is a left-invariant metric on G, which descends to a Riemannian metric on $\Gamma\diagdown G$ that is also denoted by g. A Heisenberg manifold is a two-step Riemannian nilmanifold whose covering Lie group G is one of the $(2n+1)$-dimensional Heisenberg Lie groups (see, for example, [20]). The study of nilmanifolds and nilpotent Lie groups has long been relevant to inverse spectral problems (see [19] for a survey). Nilmanifolds play an important role in the study of Dirac eigenvalues, as was shown in a paper of Ammann and C. Sprouse (see [2]). They show that if a Riemannian spin^c manifold with bounded sectional curvature and finite diameter has scalar curvature bounded from below by a sufficiently small negative number and if the smallest Dirac eigenvalue λ is sufficiently close to zero, then the manifold is diffeomorphic to a nilmanifold.

In this paper, we prove several new results concerning the computation of the eta invariant on any closed manifold. In Section 2.1, we discuss the interesting relationships between the zeta and eta functions of operators, the main point being Proposition 1, where a simple but general argument is used to show the formula

$$\frac{\partial}{\partial c}\eta_c\left(s\right) = -s\zeta_{(D+c)^2}\left(\frac{s+1}{2}\right),$$

where η_c is the eta function corresponding to the operator $D+c = D+c\mathbf{1}$, where c is a real number, and where $\zeta_{(D+c)^2}$ is the zeta function corresponding to the operator $(D+c)^2$. From this we see that changes in the eta invariant of an elliptic first order operator on a closed, odd-dimensional manifold is related to a particular residue of a pole of the zeta function corresponding to the second order operator $(D+c)^2$. This residue is, up to a constant, a coefficient in the asymptotic expansion of the trace of the heat operator $\exp\left(-t\left(D+c\right)^2\right)$. In Section 2.2, this coefficient is computed explicitly as a function of c.

Using these general results about $\frac{\partial}{\partial c}\eta_c\left(0\right)$, if $\eta_c\left(0\right)$ is known at a single value of c, the heat kernel asymptotic formula and knowledge of small eigenvalues determine $\eta_0\left(0\right)$, the eta

invariant of D. In Theorem 5, we prove a general formula for the eta invariant of a Dirac-type operator on a closed manifold in the case that the spectrum of the operator is symmetric about a certain real number $\overline{\lambda}$. We deduce from this formula a more specific formula for three-manifolds in Section 2.4, which calculates the eta invariant in terms of the volume, the total scalar curvature, the total trace of the twisting curvature, and small eigenvalues of the Dirac-type operator (notation defined in that section):

$$\begin{aligned}\eta(0) &= -\frac{\widehat{n}\overline{\lambda}^3}{6\pi^2}\mathrm{vol}(M) + \frac{\overline{\lambda}}{4\pi^2}\left(\frac{\widehat{n}}{12}\int_M \mathrm{Scal} + \int_M Tr\left(F^W\right)\right) \\ &\quad + \mathrm{sgn}\left(\overline{\lambda}\right)\left(2\#\left(\sigma(D)\cap\left(0,\overline{\lambda}\right)\right) + \#\left(\sigma(D)\cap\left\{0,\overline{\lambda}\right\}\right)\right).\end{aligned}$$

Using Kirillov theory, the spin^c Dirac operator on two-step nilmanifolds is decomposed explicitly in terms of irreducible subspaces of the right quasi-regular representation in Section 3.2. To that end, occurrence and multiplicity conditions for Dirac eigenspinors are developed in Section 3.3 in analogy to Pesce's known work [26] concerning the Laplacian. It is here that we utilize analogues of the work of C.C. Moore [25] and L. Richardson [28], developed in the appendix, Section 7. Explicit formulas for the Dirac operator are computed in terms of a special basis of spinors for each invariant subspace.

For general Heisenberg three-manifolds, the spectrum of the spin^c Dirac operator and the eta invariant are computed in terms of the metric, the lattice and spin structure in Section 5.2. The formula for the eta invariant has the form

$$\eta(0) = \frac{r^2 m_v}{96\pi^2 A^2} - N(A, r, w_2, m_v, m_w, \varepsilon),$$

where $N(A, r, w_2, m_v, m_w, \varepsilon)$ is a nonnegative integer specified in terms of $A, r; w_2, m_v, m_w; \varepsilon$, the metric, lattice, and spin structure data. In this section, we exhibit continuous families of geometrically, spectrally different Heisenberg three-manifolds whose spin^c Dirac operators have constant eta invariant. Computations for a general Heisenberg nilmanifold are done in Section 5.3; in particular, we show how to calculate the Dirac spectrum for any example. We show in Section 5.4 that the Dirac spectrum is symmetric about zero in dimensions $4m' + 1$ ($m' \in \mathbb{Z}_{>0}$). Thus, the eta invariant is automatically zero in these dimensions. In Section 6, we compute the Dirac operator of a particular five-dimensional non-Heisenberg nilmanifold, and we show that the techniques used in previous sections do not yield explicit formulas for the eigenvalues in this case.

2. The eta invariant

2.1. **Eta and zeta functions of perturbed operators.** In this section and throughout the paper, we will often use the notation $(D + c)$ for an operator, where D is an operator and c is a scalar, and we regard c in this expression as c times the identity. We also use the notation $\sigma(D)$ to denote the spectrum of D, with multiplicities.

Proposition 1. *Let D be any self-adjoint operator for which $\eta(s)$ is defined and analytic at $s = 0$. Suppose in addition that there exists an interval $I \subset \mathbb{R}$ and a constant $B > 0$ such that for all $c \in I$,*

(1) $\sum_\lambda \mathrm{sgn}(\lambda + c)\,|\lambda + c|^{-s}$ *and* $\sum_\lambda \left((\lambda + c)^2\right)^{-\frac{s+1}{2}}$ *converge absolutely for* $\mathrm{Re}(s) > B$, *and*
(2) $-c$ *is not an eigenvalue of D.*

Then the eta function $\eta_c(s)$ corresponding to the operator $D+c$ satisfies, on its domain,

$$\frac{d}{dc}\eta_c(s) = -s\zeta_{(D+c)^2}\left(\frac{s+1}{2}\right),$$

where $\zeta_{(D+c)^2}$ is the zeta function corresponding to the nonnegative operator $(D+c)^2$, that is

$$\zeta_{(D+c)^2}(s) = \sum_{\mu>0} \mu^{-s},$$

where the sum is over all positive eigenvalues with multiplicity $\{\mu\}$ of the operator $(D+c)^2$. In particular, if D is a first-order, elliptic, essentially self-adjoint differential operator, then $\frac{d}{dc}\eta_c(0)$ is the residue of the simple pole of the meromorphic function $\zeta_{(D+c)^2}\left(\frac{s+1}{2}\right)$ at $s=0$. (If $\zeta_{(D+c)^2}\left(\frac{s+1}{2}\right)$ is regular at $s=0$, then $\frac{d}{dc}\eta_c(0)=0$.)

Remark: It is known that second-order essentially self-adjoint elliptic differential operators such as $(D+c)^2$ on a manifold of dimension n yield zeta functions with at most simple poles, and they are located at $s=\frac{n}{2}$, $s=\frac{n}{2}-1$, $s=\frac{n}{2}-2$, ... for n odd and at $s=\frac{n}{2}$, $s=\frac{n}{2}-1$, ... , $s=1$ for n even. See [16] for specifics. Further, the residues at these poles are given by explicitly computable integrals of locally-defined functions.

Proof. We know that for each eigenvalue λ of D, $sgn(\lambda+c)$ does not vary with $c \in I$. Then for large $\mathrm{Re}(s)$,

$$\begin{aligned}
\eta_c(s) &= \sum_\lambda \mathrm{sgn}(\lambda+c)\left((\lambda+c)^2\right)^{-s/2} \\
\frac{d}{dc}\eta_c(s) &= \sum_\lambda \mathrm{sgn}(\lambda+c)\left(-\frac{s}{2}\left((\lambda+c)^2\right)^{-s/2-1}\right)2(\lambda+c) \\
&= -s\sum_\lambda \mathrm{sgn}(\lambda+c)\,|\lambda+c|^{-s-2}(\lambda+c) \\
&= -s\sum_\lambda \left((\lambda+c)^2\right)^{-\frac{s+1}{2}} = -s\zeta_{(D+c)^2}\left(\frac{s+1}{2}\right).
\end{aligned}$$

Since both sides are analytic in s for large $\mathrm{Re}(s)$, the statement must remain true after analytic continuation. □

We are interested in the eta invariant, which is $\eta_c(0)$. By the formula in the proposition above, the relevant information is the residue of the pole of the zeta function $\zeta_{(D+c)^2}(z)$ at $z=\frac{1}{2}$. For odd-dimensional manifolds, this is a constant times one of the heat invariants. If the manifold is even-dimensional, there is no pole at $z=\frac{1}{2}$, so that $\frac{d}{dc}\eta_c(0)=0$.

Corollary 2. *If the manifold is even-dimensional, then $\frac{d}{dc}\eta_c(0)=0$, so that the eta invariant is constant with respect to c on intervals where $D+c$ has trivial kernel, and then it changes by integral jumps in general.*

We also have the following result about perturbations of zeta functions.

Proposition 3. *With the assumptions of Proposition 1,*

$$\frac{d}{dc}\zeta_{(D+c)^2}(s) = -2s\eta_c(2s+1).$$

Proof. For large $\operatorname{Re}(s)$,

$$\begin{aligned}\frac{d}{dc}\zeta_{(D+c)^2}(s) &= \frac{d}{dc}\sum_{\lambda}\left((\lambda+c)^2\right)^{-s} \\ &= \sum_{\lambda}-s\left((\lambda+c)^2\right)^{-s-1}2(\lambda+c) = -2s\sum_{\lambda}|\lambda+c|^{-2s-2}(\lambda+c) \\ &= -2s\sum_{\lambda}\operatorname{sgn}(\lambda+c)|\lambda+c|^{-2s-1} = -2s\eta_c(2s+1).\end{aligned}$$

Since both sides are analytic in s for large $\operatorname{Re}(s)$, the statement must remain true after analytic continuation. □

2.2. **Heat Kernel Asymptotics.** Because of Proposition 1, we will be interested in the residues of $\zeta_{(D+c)^2}(s)$ at its poles, which are determined by the heat kernel asymptotics (see Section 2.3). Specifically, we need the asymptotics as $t \to 0^+$ of

$$Tr\left(\exp\left(-t(D+c)^2\right)\right) = \int_M Tr\ K_c(t,x,x)\ d\mathrm{vol},$$

where we assume $D = \sum(e_j\diamond)\nabla_{e_j} : C^\infty(E) \to C^\infty(E)$ is a Dirac-type operator and $c \in \mathbb{R}$. We will let n be the dimension of the manifold M, and we will let $\widehat{n}$ be the rank of the vector bundle E. Here and in what follows, we use the $\diamond$ symbol to denote Clifford multiplication. The element $K_c(t,x,x) \in \operatorname{End}(E_x)$ is

$$K_c(t,x,x) = e^{-t(D+c)^2}(x,x),$$

which satisfies

$$\begin{aligned}\left(\frac{\partial}{\partial t}+(D+c)^2\right)K_c(t,x,y) &= 0 \\ \lim_{t\to 0^+}K_c(t,x,y) &= \delta_{xy},\end{aligned}$$

where δ_{xy} is the Dirac delta distribution. To find the asymptotics as $t \to 0^+$, we need to solve for $u_k(x,y) \in \operatorname{Hom}(E_y, E_x)$, where

$$K_c(t,x,y) \sim \frac{1}{(4\pi t)^{n/2}}e^{-r^2/4t}\left(u_0(x,y)+tu_1(x,y)+t^2u_2(x,y)+\ldots\right) \tag{1}$$

where $r = \operatorname{dist}(x,y)$. Such an asymptotic expansion exists, since $(D+c)^2$ is a generalized Laplacian (see [6], [16], [29]).

We will assume that we have chosen geodesic normal coordinates $x = (x_1,\ldots,x_n)$ centered at $y = 0$ and that the frame field $(e_1,\ldots,e_n)$ is parallel translated radially from the origin (i.e. y) such that

$$e_j(0) = \partial_j\ .$$

Then in these coordinates, we may map E_x to E_y via radial parallel translation, so that each $u_k(x,y)$ may be regarded as a matrix-valued function of x, with $\mathbb{R}^{\widehat{n}}$ identified with E_y. Observe that the Dirac operator may be expressed as

$$D = \sum_j e_j\diamond\nabla_{e_j} = \sum_{p,q}g^{pq}\partial_p\diamond\nabla_{\partial_q}\ ,$$

where in the first case we are summing over an orthonormal frame, and in the second case we are using the coordinate vector fields, with (g^{pq}) the inverse of the metric matrix (g_{ij}).

We have, using the Einstein summation convention,

$$\begin{aligned}
-(D+c)^2 &= -(e_i \diamond \nabla_{e_i} + c)^2 \\
&= -(e_i \diamond \nabla_{e_i})\left(e_j \diamond \nabla_{e_j}\right) - 2c\,(e_i \diamond \nabla_{e_i}) - c^2 \\
&= -(e_i\diamond)(e_j\diamond)\nabla_{e_i}\nabla_{e_j} + \left[-(e_i\diamond)\left((\nabla_{e_i} e_j)\diamond\right) - 2c\,(e_j\diamond)\right]\nabla_{e_j} - c^2 \\
&= \nabla_{e_i}\nabla_{e_i} - \sum_{i<j}(e_i\diamond)(e_j\diamond)\left[\nabla_{e_i},\nabla_{e_j}\right] + \left[-(e_i\diamond)\left((\nabla_{e_i} e_j)\diamond\right) - 2c\,(e_j\diamond)\right]\nabla_{e_j} - c^2 \\
&= \nabla_{e_i}\nabla_{e_i} - \sum_{i<j}(e_i\diamond)(e_j\diamond)\left(\nabla_{[e_i,e_j]}\right) + \left[-(e_i\diamond)\left((\nabla_{e_i} e_j)\diamond\right) - 2c\,(e_j\diamond)\right]\nabla_{e_j} \\
&\quad - \sum_{i<j}\underbrace{(e_i\diamond)(e_j\diamond)\left(\left[\nabla_{e_i},\nabla_{e_j}\right] - \nabla_{[e_i,e_j]}\right)}_{\text{define this to be } K_{ij}} - c^2. \qquad (2)
\end{aligned}$$

Further, let $K = \sum_{i<j} K_{ij} \in \mathrm{End}\,(E_x)$.

Next, let s be a bundle endomorphism, and let f be any function. Let $h = \frac{1}{(4\pi t)^{n/2}} e^{-r^2/4t}$, and let $g = \det\,(g_{ij})$, where r is the geodesic distance to $y = 0$. Then from the formulas in [29, pp. 99-100] (extended, as is common, to endomorphisms),

$$\begin{aligned}
\nabla h &= -\frac{h}{2t} r\partial_r \\
\frac{\partial h}{\partial t} + \Delta h &= \frac{rh\partial_r g}{4gt} \\
D\,(fs) - fDs &= (\nabla f)\diamond s \\
D^2\,(fs) - fD^2 s &= (\Delta f)\, s - 2\nabla_{\nabla f} s,
\end{aligned}$$

so

$$\begin{aligned}
\left(-(D+c)^2\right)(fs) &= -\left(D^2 + 2cD + c^2\right)(fs) \\
&= -\left(fD^2 s + (\Delta f)\, s - 2\nabla_{\nabla f} s\right) - 2c\,(fDs + (\nabla f)\diamond s) - c^2 fs \\
&= -f\,(D+c)^2 s - (\Delta f)\, s + 2\nabla_{\nabla f} s - 2c\,(\nabla f)\diamond s.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{1}{h}\left(\partial_t + (D+c)^2\right)(hs) &= \left(-\frac{1}{h}\Delta h + \frac{r\partial_r g}{4gt}\right) s + \partial_t s + (D+c)^2 s \\
&\quad + \left(\frac{\Delta h}{h}\right) s - \frac{2}{h}\nabla_{\nabla h} s + \frac{2c}{h}(\nabla h)\diamond s \\
&= \partial_t s + (D+c)^2 s + \frac{r}{4gt}\partial_r g s + \frac{1}{t}\nabla_{r\partial_r} s - \frac{c}{t}(r\partial_r)\diamond s.
\end{aligned}$$

Writing

$$s = u_0 + tu_1 + t^2 u_2 + ...,$$

we solve $\left(\partial_t + (D+c)^2\right)(hs) = 0$ and get the equations

$$\nabla_{r\partial_r} u_j + \left(j + \frac{r\partial_r g}{4g} - c\,(r\partial_r\diamond)\right) u_j = -(D+c)^2 u_{j-1}\ , \qquad (3)$$

or

$$\nabla_{\partial_r} u_j + \left(\frac{j}{r} + \frac{\partial_r g}{4g} - c\left(\partial_r \diamond\right)\right) u_j = -\frac{1}{r}\left(D+c\right)^2 u_{j-1} \tag{4}$$

This is an ordinary differential equation along a geodesic emanating from y, the center of the geodesic coordinates.

Note that for any smooth function f,

$$\begin{aligned}
\exp\left(f\left(r\right)\left(\partial_r \diamond\right)\right) &= \sum_{k\geq 0} \frac{1}{(2k)!} f\left(r\right)^{2k} \left(\partial_r \diamond\right)^{2k} + \sum_{k\geq 0} \frac{1}{(2k+1)!} f\left(r\right)^{2k+1} \left(\partial_r \diamond\right)^{2k+1} \\
&= \sum_{k\geq 0} \frac{(-1)^k}{(2k)!} f\left(r\right)^{2k} + \left(\sum_{k\geq 0} \frac{(-1)^k}{(2k+1)!} f\left(r\right)^{2k+1}\right)\left(\partial_r \diamond\right) \\
&= \cos\left(f\left(r\right)\right)\mathbf{1} + \sin\left(f\left(r\right)\right)\left(\partial_r \diamond\right).
\end{aligned}$$

We also have the operator equation

$$\begin{aligned}
& \nabla_{\partial_r}\left[\cos\left(f\left(r\right)\right)\mathbf{1} + \sin\left(f\left(r\right)\right)\left(\partial_r \diamond\right)\right] \\
= \;& -f'\left(r\right)\sin\left(f\left(r\right)\right)\mathbf{1} + \cos\left(f\left(r\right)\right)\nabla_{\partial_r} + f'\left(r\right)\cos\left(f\left(r\right)\right)\left(\partial_r \diamond\right) + \sin\left(f\left(r\right)\right)\left(\partial_r \diamond\right)\nabla_{\partial_r} \\
= \;& \left[\cos\left(f\left(r\right)\right)\mathbf{1} + \sin\left(f\left(r\right)\right)\left(\partial_r \diamond\right)\right]\nabla_{\partial_r} + -f'\left(r\right)\sin\left(f\left(r\right)\right)\mathbf{1} + f'\left(r\right)\cos\left(f\left(r\right)\right)\left(\partial_r \diamond\right) \\
= \;& \left[\cos\left(f\left(r\right)\right)\mathbf{1} + \sin\left(f\left(r\right)\right)\left(\partial_r \diamond\right)\right]\left(\nabla_{\partial_r} + f'\left(r\right)\left(\partial_r \diamond\right)\right).
\end{aligned}$$

Thus we multiply (4) by $r^j g^{1/4}\left[\cos\left(-cr\right)\mathbf{1} + \sin\left(-cr\right)\left(\partial_r \diamond\right)\right]$. Then observe that

$$\begin{aligned}
& \nabla_{\partial_r}\left(r^j g^{1/4}\left[\cos\left(-cr\right)\mathbf{1} + \sin\left(-cr\right)\left(\partial_r \diamond\right)\right] u_j\right) \\
= \;& r^j g^{1/4}\left[\cos\left(-cr\right)\mathbf{1} + \sin\left(-cr\right)\left(\partial_r \diamond\right)\right]\left(\nabla_{\partial_r} + \left(\frac{j}{r} + \frac{\partial_r g}{4g} - c\left(\partial_r \diamond\right)\right)\right) u_j \\
= \;& -\frac{1}{r} r^j g^{1/4}\left[\cos\left(-cr\right)\mathbf{1} + \sin\left(-cr\right)\left(\partial_r \diamond\right)\right]\left(D+c\right)^2 u_{j-1},
\end{aligned}$$

so the new recursion formula is

$$\begin{aligned}
& \nabla_{\partial_r}\left(r^j g^{1/4}\left[\cos\left(-cr\right)\mathbf{1} + \sin\left(-cr\right)\left(\partial_r \diamond\right)\right] u_j\right) \\
= \;& -r^{j-1} g^{1/4}\left[\cos\left(-cr\right)\mathbf{1} + \sin\left(-cr\right)\left(\partial_r \diamond\right)\right]\left(D+c\right)^2 u_{j-1}\,.
\end{aligned} \tag{5}$$

Substituting $j=0$, we see that $g^{1/4}\left[\cos\left(-cr\right)\mathbf{1} + \sin\left(-cr\right)\left(\partial_r \diamond\right)\right] u_0$ is parallel along radial geodesics, which means that

$$\begin{aligned}
u_0\left(r\right) &= g^{-1/4}\left[\cos\left(-cr\right)\mathbf{1} - \sin\left(-cr\right)\left(\partial_r \diamond\right)\right] \\
&= g^{-1/4}\left[\cos\left(cr\right)\mathbf{1} + \sin\left(cr\right)\left(\partial_r \diamond\right)\right].
\end{aligned} \tag{6}$$

In other words, $u_0\left(r\right)$ is the linear map from E_y to E_x (with y being the origin of the geodesic coordinate system and r being the distance from y to x) defined by

$$s\left(y\right) \mapsto g^{-1/4}\left[\cos\left(cr\right)\mathbf{1} + \sin\left(cr\right)\left(\partial_r \diamond\right)\right] s\left(x\right),$$

where $s\left(x\right)$ is the radial parallel translate of $s\left(y\right)$ along the geodesic connecting y to x.

By writing

$$u_1 = u_1\left(0\right) + \mathcal{O}\left(r\right),$$

from (3) we see

$$u_1 + r\left(\nabla_{\partial_r} u_1 + \left(\frac{\partial_r g}{4g} - c\left(\partial_r \diamond\right)\right) u_1\right) = -\left(D+c\right)^2 u_0.$$

In particular,

$$u_1\left(0\right) = \left(-\left(D+c\right)^2 u_0\right)\left(0\right).$$

Since $r^2 = x_j x_j$, $r\partial_r = x_j\partial_j$, and $g = 1+\frac{1}{3}R_{ipqi}x_p x_q + \mathcal{O}\left(r^3\right)$ in geodesic normal coordinates in terms of the Riemann curvature tensor $R_{ijk\ell}$ at $x=0$ (see [29, p. 104]), using the binomial expansion,

$$\begin{aligned} u_0 &= g^{-1/4}\left[\cos\left(cr\right)\mathbf{1} + \sin\left(cr\right)\left(\partial_r\diamond\right)\right] \\ &= \mathbf{1}+cr\left(\partial_r\diamond\right) - \frac{c^2r^2}{2}\mathbf{1} - \frac{1}{12}R_{ijki}x_jx_k\mathbf{1}+\mathcal{O}\left(r^3\right) \\ &= \mathbf{1}+cx_j\left(\partial_j\diamond\right) - \frac{c^2x_jx_j}{2}\mathbf{1} - \frac{1}{12}R_{ijki}x_jx_k\mathbf{1}+\mathcal{O}\left(r^3\right). \end{aligned}$$

Then at 0,

$$\begin{aligned} \left(Du_0\right)\left(0\right) &= g^{pq}\left(\partial_p\diamond\right)\nabla_{\partial_q}u_0 \\ &= \left(\partial_p\diamond\right)\nabla_{\partial_p}u_0 \\ &= \left(\partial_p\diamond\right)c\left(\partial_p\diamond\right) = -nc\mathbf{1}. \end{aligned}$$

At 0, $\nabla_{\partial_p}\partial_q = 0$ for all p,q ; thus, from (2) and the above,

$$\begin{aligned} \left(D^2u_0\right)\left(0\right) &= \left(-\nabla_{\partial_p}\nabla_{\partial_p} + K\right)u_0 \\ &= \left(nc^2+\frac{1}{6}R_{ijji} + K\right)\mathbf{1} = \left(nc^2 - \frac{1}{6}\mathrm{Scal} + K\right)\mathbf{1}, \end{aligned}$$

where Scal denotes the scalar curvature. Then

$$\begin{aligned} u_1\left(0\right) &= \left(-\left(D^2+2cD+c^2\right)u_0\right)\left(0\right) \\ &= -\left(nc^2 - \frac{1}{6}\mathrm{Scal} + K - 2cnc + c^2\right)\mathbf{1} \\ &= \left(\left(n-1\right)c^2 + \frac{1}{6}\mathrm{Scal}\right)\mathbf{1} - K. \end{aligned} \tag{7}$$

We have shown that the heat kernel for $\left(D+c\right)^2$ has the expansion

$$\begin{aligned} K_c\left(t,x,x\right) &:= \exp\left(-t\left(D+c\right)^2\right)\left(x,x\right) \\ &= \frac{1}{\left(4\pi t\right)^{n/2}}\left(\mathbf{1} + t\left(\left(\left(n-1\right)c^2 + \frac{1}{6}\mathrm{Scal}\right)\mathbf{1} - K\right) + \mathcal{O}\left(t^2\right)\right), \end{aligned}$$

$$\begin{aligned} \mathrm{Tr}\exp\left(-t\left(D+c\right)^2\right) = \frac{1}{\left(4\pi t\right)^{n/2}}\Bigg(&\widehat{n}\mathrm{vol}\left(M\right) \\ &+ t\left[\widehat{n}\left(n-1\right)c^2\mathrm{vol}\left(M\right) + \frac{\widehat{n}}{6}\int_M \mathrm{Scal} - \int_M Tr\left(K\right)\right] + \mathcal{O}\left(t^2\right)\Bigg). \end{aligned}$$

Here, n is the dimension of the manifold, and $\widehat{n}$ is the rank of the bundle E.

Note that on a spin manifold, the Clifford contracted curvature term K has the form (see [29, pp. 48–49])

$$K = \frac{\mathrm{Scal}}{4} + F^{E/S},$$

where S is the spin^c bundle, $E = S \otimes W$ with connection $\nabla^{S\otimes W} = \nabla^W \otimes 1 + 1 \otimes \nabla^S$, and $F^{E/S}$ is the twisting curvature of E, meaning

$$F^{E/S} = F^W = \sum_{i<j} F^W(e_i, e_j)\left(e^i\diamond\right)\left(e^j\diamond\right),$$

with F^W the curvature of ∇^W. In particular, if D is the spin^c Dirac operator on a spin manifold, then $F^W = 0$ and

$$\operatorname{Tr}\exp\left(-t\,(D+c)^2\right) = \frac{1}{(4\pi t)^{n/2}}\left(\widehat{n}\mathrm{vol}\,(M) + t\left[\widehat{n}\,(n-1)\,c^2\mathrm{vol}\,(M) - \frac{\widehat{n}}{12}\int_M \mathrm{Scal}\right] + \mathcal{O}\left(t^2\right)\right).$$

Observe that our first recursion formula (4) for the heat invariant endomorphism u_j corresponding to $(D+c)^2$ is

$$\nabla_{\partial_r} u_j + \left(\frac{j}{r} + \frac{\partial_r g}{4g} - c\,(\partial_r\diamond)\right)u_j = -\frac{1}{r}\,(D+c)^2\,u_{j-1}\ ,$$

where r is the distance from the origin of the coordinate system, and the differential equation holds along a geodesic from 0 to x. For $j \geq 0$, we expand

$$u_j = \sum_{k=0}^{K} c^k u_{j,k}\ + \mathcal{O}\left(c^{K+1}\right),$$

where each $u_{j,k}$ is independent of $c \in \mathbb{R}$. For consistency we declare that $u_{j,k} = 0$ if either j or k is negative. Our recursive formula above implies that (collecting powers of c)

$$\nabla_{\partial_r} u_{j,k} + \left(\frac{j}{r} + \frac{\partial_r g}{4g}\right)u_{j,k} = (\partial_r\diamond)\,u_{j,(k-1)} - \frac{1}{r}D^2 u_{(j-1),k} - \frac{2}{r}D u_{(j-1),(k-1)} - \frac{1}{r}u_{(j-1),(k-2)}. \quad (8)$$

Proposition 4. *We have*

$$u_{j,k} = \mathcal{O}\left(r^{\max\{k-2j,0\}}\right).$$

In particular,

$$u_{j,k}(0) = 0$$

if $k > 2j$, so that u_j is a polynomial in c of degree at most $2j$.

Proof. Clearly, $u_{j,0} = \mathcal{O}(1)$ for all $j \geq 0$, as these refer to the standard heat invariants (with $c = 0$). Also, the formula holds for $u_{0,k}$ by Taylor analysis of the expicit formula (6). We prove the general case by induction; assume that the theorem holds for all (j,k) such that $0 \leq j < J$ and $k \geq 0$ or $j = J$ and $0 \leq k \leq K$, with $J \geq 1$ and $K \geq 0$. Then the formula preceding the statement implies that

$$r\nabla_{\partial_r} u_{J,K+1} + \left(J + \frac{r\partial_r g}{4g}\right)u_{J,K+1} = r\,(\partial_r\diamond)\,u_{J,K} - D^2 u_{(J-1),(K+1)} - 2Du_{(J-1),K} - u_{(J-1),(K-1)}\ .$$

Note that, given $A(r) = \mathcal{O}(r^p)$ is smooth in r, we have $r\partial_r A(r) = \mathcal{O}(r^p)$ if $p \neq 0$ and $r\partial_r A(r) = \mathcal{O}(r)$ if $p = 0$. Similarly, $r(\partial_r \diamond) A(r) = \mathcal{O}(r^{p+1})$ since $\partial_r \diamond$ is bounded and has constant norm. Then, by the induction hypothesis,

$$\begin{aligned} u_{J,K+1} &= \mathcal{O}\left(r^{\max\{K-2J,0\}+1}\right) + \mathcal{O}\left(r^{\max\{K-2J+3-2,0\}}\right) \\ &\quad + \mathcal{O}\left(r^{\max\{K-2J+2-1,0\}}\right) + \mathcal{O}\left(r^{\max\{K-2J+1,0\}}\right) \\ &= \mathcal{O}\left(r^{\max\{K-2J+1,0\}}\right), \end{aligned}$$

since $D(\mathcal{O}(r^p)) = \mathcal{O}\left(r^{\max\{p-1,0\}}\right)$ as long as the quantities are smooth in r. □

Because $(\partial_r \diamond)^{2j} = (-1)^j$ and $(\partial_r \diamond)^{2j+1} = (-1)^j (\partial_r \diamond)$, from (6) we have

$$u_{0,k} = \frac{1}{k!} g^{-1/4} r^k (\partial_r \diamond)^k .$$

Also, since all of the $u_{j,0}$ are known (the standard heat invariants), we may use (8) to calculate $u_{j,k}$ for all $j \geq 0$, $k \geq 0$. That is,

$$\nabla_{\partial_r}\left(r^j g^{1/4} u_{j,k}\right) = r^j g^{1/4}\left((\partial_r \diamond) u_{j,(k-1)} - \frac{1}{r} D^2 u_{(j-1),k} - \frac{2}{r} D u_{(j-1),(k-1)} - \frac{1}{r} u_{(j-1),(k-2)}\right),$$

and so the expression may be integrated along a radial geodesic to solve for $u_{j,k}$. From the formulas for $u_{0,k}$ and (7) we have

$$u_{0,0}(0) = 1,\ u_{1,0}(0) = \left(\frac{1}{6}\mathrm{Scal}\right)\mathbf{1} - K,\ u_{1,1}(0) = 0,\ u_{1,2}(0) = (n-1)\mathbf{1}.$$

Let

$$a_{j,k} = \int_M \mathrm{tr}\left(u_{j,k}(x,x)\right)\ \mathrm{dvol}\ ,$$

where $u_{j,k}(x,x)$ is the expression at $r = 0$ of $u_{j,k}$ found above. In particular, if n is the dimension of the manifold M and $\widehat{n}$ is the rank of the bundle E,

$$a_{0,0} = \widehat{n}\mathrm{vol}(M),\ a_{1,0} = \frac{\widehat{n}}{6}\int_M \mathrm{Scal} - \int_M Tr(K),\ a_{1,1} = 0,\ a_{1,2} = \widehat{n}(n-1)\mathrm{vol}(M). \quad (9)$$

Then the heat invariants $a_j(c)$ corresponding to $(D+c)^2$ satisfy

$$a_j(c) = \int_M \mathrm{tr}\left(u_j(x,x)\right)\ \mathrm{dvol} = \sum_{k=0}^{2j} c^k a_{j,k}\ . \quad (10)$$

2.3. **The eta invariant for arbitrary manifolds with spectral symmetry .** Suppose that M is a closed Riemannian manifold of dimension n. Recall from Proposition 1, we wish to calculate $\lim_{s\to 0} -s\zeta_{(D+c)^2}\left(\frac{s+1}{2}\right)$, at a particular value of c where $\dim\ker(D+c)^2 = \{0\}$. From (1), as $t \to 0^+$,

$$\sum_\mu e^{-t\mu} = \int_M \mathrm{tr}K_c(t,x,x)\ dV(x) \sim \frac{1}{(4\pi t)^{n/2}}\left(a_0 + ta_1 + t^2 a_2 + ...\right),$$

where $\{\mu\}$ are the eigenvalues of $(D+c)^2$ with multiplicities. The standard derivation of the analytic continuation of the zeta function is as follows. For large $\operatorname{Re}(s)$,

$$\begin{aligned}
\zeta_{(D+c)^2}(s) &= \sum_\mu \mu^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left(\sum_\mu e^{-t\mu}\right)dt \\
&= \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(\frac{1}{(4\pi t)^{n/2}}\left(a_0+a_1t+\dots+a_Nt^N\right)\right)dt \\
&\quad+\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(\sum e^{-t\mu}-\frac{1}{(4\pi t)^{n/2}}\left(a_0+a_1t+\dots+a_Nt^N\right)\right) \\
&\quad+\frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}\left(\sum e^{-t\mu}\right)dt
\end{aligned}$$

$$=\frac{1}{(4\pi)^{n/2}\Gamma(s)}\sum_{j=0}^N a_j\int_0^1 t^{s-1-\frac{n}{2}+j}dt+\phi_N(s)=\frac{1}{(4\pi)^{n/2}\Gamma(s)}\sum_{j=0}^N\frac{a_j}{s-\frac{n}{2}+j}+\phi_N(s),$$

where $\phi_N(s)$ is holomorphic for $\operatorname{Re}s>\frac{n}{2}-N-1$, $\Gamma(\cdot)$ is the Gamma function, and a_j is the heat invariant corresponding to $(D+c)^2$:

$$a_j=\int_M \operatorname{tr}\left(u_j(x,x)\right)\,\mathrm{dvol}\,.$$

Then, since $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$,

$$\lim_{s\to 0}-s\zeta_{(D+c)^2}\left(\frac{s+1}{2}\right)=\lim_{s\to 0}\frac{-s}{(4\pi)^{n/2}\Gamma\left(\frac{s+1}{2}\right)}\frac{a_{\frac{n-1}{2}}}{\left(\frac{s+1}{2}-\frac{1}{2}\right)}=-2^{1-n}\pi^{-(n+1)/2}a_{\frac{n-1}{2}},$$

or

$$\frac{d}{dc}\eta_c(0)=-2^{1-n}\pi^{-(n+1)/2}a_{\frac{n-1}{2}}(c)\,.$$

Note that if n is even, $\frac{d}{dc}\eta_c(0)=0$.

Now, suppose that there is a point of symmetry, $\overline{\lambda}<0$, in the spectrum $\sigma(D)$ of D, meaning that $\sigma(D)-\overline{\lambda}$ is symmetric about 0 in $\mathbb{R}$. Then $\eta_{-\overline{\lambda}}(0)=0$. We then integrate the formula above from $c=0$ to $c=-\overline{\lambda}$. We have a discontinuity (a jump of $+2$) at each $c\in\left(0,-\overline{\lambda}\right)$ that is an eigenvalue of $-D$, due to the fact that $c\mapsto\operatorname{sgn}(\lambda+c)$ has a similar discontinuity near $c=-\lambda$. Also, if either 0 or $-\overline{\lambda}$ are contained in the spectrum of $-D$, then we will have a jump discontinuity of $+1$ at those points. Let $c_1\leq\dots\leq c_k$ be the points of $\left(0,-\overline{\lambda}\right)$ that are eigenvalues of $-D$. Let $n_0=\begin{cases}1 & \text{if } 0\in\sigma(D)\\ 0 & \text{otherwise}\end{cases}$, $n_{-\overline{\lambda}}=$

$\begin{cases} 1 & \text{if } \overline{\lambda} \in \sigma(D) \\ 0 & \text{otherwise} \end{cases}$. Then the fundamental theorem of calculus yields

$$\begin{aligned} \int_0^{c_1} \frac{d}{dc}\eta_c(0)\ dc &= \eta_{c_1}(0) - \eta_0(0) - 1 - n_0, \\ \int_{c_j}^{c_{j+1}} \frac{d}{dc}\eta_c(0)\ dc &= \eta_{c_{j+1}}(0) - \eta_{c_j}(0) - 2, \\ \int_{c_k}^{-\overline{\lambda}} \frac{d}{dc}\eta_c(0)\ dc &= \eta_{-\overline{\lambda}}(0) - \eta_k(0) - 1 - n_{-\overline{\lambda}}, \end{aligned}$$

which add to

$$\int_0^{-\overline{\lambda}} \frac{d}{dc}\eta_c(0)\ dc = \eta_{-\overline{\lambda}}(0) - \eta_0(0) - 2k - n_0 - n_{-\overline{\lambda}}.$$

Therefore, since $\eta_{-\overline{\lambda}}(0) = 0$ and $\eta_0(0) = \eta(0)$,

$$\begin{aligned} \eta(0) &= -\int_0^{-\overline{\lambda}} \frac{d}{dc}\eta_c(0)\ dc - 2k - n_0 - n_{-\overline{\lambda}} \\ &= \int_{-\overline{\lambda}}^0 \frac{d}{dc}\eta_c(0)\ dc - 2k - n_0 - n_{-\overline{\lambda}} \end{aligned}$$

In the case where the point of symmetry is positive ($\overline{\lambda} > 0$), the calculation above may be adapted in the following ways. We integrate the formula for $\frac{d}{dc}\eta_c(0)$ from $c = -\overline{\lambda}$ to $c = 0$, and if $c_1 \leq ... \leq c_k$ are the points of $\left(-\overline{\lambda}, 0\right)$ that are eigenvalues of $-D$, we have

$$\begin{aligned} \int_{-\overline{\lambda}}^{c_1} \frac{d}{dc}\eta_c(0)\ dc &= \eta_{c_1}(0) - \eta_{-\overline{\lambda}}(0) - 1 - n_{-\overline{\lambda}}, \\ \int_{c_j}^{c_{j+1}} \frac{d}{dc}\eta_c(0)\ dc &= \eta_{c_{j+1}}(0) - \eta_{c_j}(0) - 2, \\ \int_{c_k}^{0} \frac{d}{dc}\eta_c(0)\ dc &= \eta_0(0) - \eta_k(0) - 1 - n_0, \end{aligned}$$

which yields

$$\eta_0(0) = \int_{-\overline{\lambda}}^0 \frac{d}{dc}\eta_c(0)\ dc + 2k + n_0 + n_{-\overline{\lambda}},$$

with $n_0,\ n_{-\overline{\lambda}}$ defined above.

In general, if $\overline{\lambda}$ is the point of symmetry of $\sigma(D)$,

$$\begin{aligned} &\eta_0(0) = \eta_{-\overline{\lambda}}(0) + \int_{-\overline{\lambda}}^0 \frac{d}{dc}\eta_c(0)\ dc + \operatorname{sgn}\left(\overline{\lambda}\right)\left(2\#\left(\sigma(D)\cap I_{\overline{\lambda}}\right) + \#\left(\sigma(-D)\cap\left\{0,-\overline{\lambda}\right\}\right)\right) \\ &= -2^{1-n}\pi^{-(n+1)/2}\int_{-\overline{\lambda}}^0 a_{\frac{n-1}{2}}(c)\ dc + \operatorname{sgn}\left(\overline{\lambda}\right)2\#\left(\sigma(D)\cap I_{\overline{\lambda}}\right) + \operatorname{sgn}\left(\overline{\lambda}\right)\#\left(\sigma(D)\cap\left\{0,\overline{\lambda}\right\}\right), \end{aligned}$$

where $I_{\overline{\lambda}}$ is $\left(0,\overline{\lambda}\right)$ or $\left(\overline{\lambda},0\right)$, depending on the sign of $\overline{\lambda}$, and where the last two terms include multiplicities.

Thus, from the formula above and (10) we have the following formula for $\eta(0) = \eta_0(0)$.

Theorem 5. *Let $\sigma(D)-\overline{\lambda}$ be symmetric about 0 in $\mathbb{R}$. Then the eta invariant satisfies*

$$\begin{aligned}\eta(0) &= -2^{1-n}\pi^{-(n+1)/2}\left(\sum_{k=0}^{n-1}\frac{(-1)^k}{k+1}\overline{\lambda}^{k+1}a_{\frac{n-1}{2},k}\right)\\ &\quad+\operatorname{sgn}\left(\overline{\lambda}\right)2\#\left(\sigma(D)\cap I_{\overline{\lambda}}\right)+\operatorname{sgn}\left(\overline{\lambda}\right)\#\left(\sigma(D)\cap\left\{0,\overline{\lambda}\right\}\right),\end{aligned}$$

where $I_{\overline{\lambda}}$ is the open interval between 0 and $\overline{\lambda}$, and where implicitly the last two terms include multiplicities.

2.4. **The zeta function and the eta invariant for three-manifolds.** By Theorem 5, for $n=3$ we have

$$\begin{aligned}\eta(0) &= -2^{-2}\pi^{-2}\left(\overline{\lambda}^{1}a_{1,0}-\frac{1}{2}\overline{\lambda}^{2}a_{1,1}+\frac{1}{3}\overline{\lambda}^{3}a_{1,2}\right)\\ &\quad+\operatorname{sgn}\left(\overline{\lambda}\right)2\#\left(\sigma(D)\cap I_{\overline{\lambda}}\right)+\operatorname{sgn}\left(\overline{\lambda}\right)\#\left(\sigma(D)\cap\left\{0,\overline{\lambda}\right\}\right),\end{aligned}$$

From (9),

$$\begin{aligned}\eta(0) &= -\frac{\widehat{n}\overline{\lambda}^{3}}{6\pi^2}\operatorname{vol}(M)-\frac{\overline{\lambda}}{4\pi^2}\left(\frac{\widehat{n}}{6}\int_M\mathrm{Scal}-\int_M Tr(K)\right)\\ &\quad+\operatorname{sgn}\left(\overline{\lambda}\right)\left(2\#\left(\sigma(D)\cap\left(0,\overline{\lambda}\right)\right)+\#\left(\sigma(D)\cap\left\{0,\overline{\lambda}\right\}\right)\right),\end{aligned}$$

where implicitly the last two terms include multiplicities. Note that every three-manifold is spin, and thus if we let F^W be the twisting curvature, then

$$\int_M Tr(K)=\int_M\frac{\widehat{n}\mathrm{Scal}}{4}+\int_M Tr\left(F^W\right).$$

Then we have

$$\begin{aligned}\eta(0) &= -\frac{\widehat{n}\overline{\lambda}^{3}}{6\pi^2}\operatorname{vol}(M)+\frac{\overline{\lambda}}{4\pi^2}\left(\frac{\widehat{n}}{12}\int_M\mathrm{Scal}+\int_M Tr\left(F^W\right)\right)\\ &\quad+\operatorname{sgn}\left(\overline{\lambda}\right)\left(2\#\left(\sigma(D)\cap\left(0,\overline{\lambda}\right)\right)+\#\left(\sigma(D)\cap\left\{0,\overline{\lambda}\right\}\right)\right).\end{aligned}\tag{11}$$

3. Two-step Nilmanifolds and Dirac operators

3.1. **Two-step Nilmanifolds and the Laplace-Beltrami operator.** We review known results about the Laplacian on two-step nilmanifolds in this section. A Lie algebra $\mathfrak{g}$ is two-step nilpotent if its derived algebra $\mathfrak{z}'=[\mathfrak{g},\mathfrak{g}]$ is contained in its nontrivial center; i.e., $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]\equiv 0$ but $[\mathfrak{g},\mathfrak{g}]\not\equiv 0$. A Lie group G is two-step nilpotent if its Lie algebra is. Let G be a simply connected two-step nilpotent Lie group of dimension n with Lie algebra $\mathfrak{g}$. Let Γ be a cocompact (i.e., $\Gamma\backslash G$ compact), discrete subgroup of G, and denote $M=\Gamma\backslash G$. Fix an inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, which corresponds to a left-invariant metric on G, and which descends to a (no longer left-invariant) Riemannian metric on M. Let $\{X_i\}$ be an orthonormal basis of left-invariant vector fields of $\mathfrak{g}$.

The existence of a cocompact, discrete subgroup Γ implies that G is unimodular, which in turn implies that the Laplace-Beltrami operator acting on smooth functions on G can be expressed as

$$\Delta=-\sum X_i^2.$$

Denote by ρ the (right) quasi-regular representation of G on $L^2(\Gamma\diagdown G)$; i.e., for $g \in G$, $f \in L^2(\Gamma\diagdown G)$,

$$(\rho(g) f)(x) = f(xg).$$

This is a unitary representation of G, and ρ is the induced representation of the trivial representation of Γ. Denote by ρ_* the associated unitary action of $\mathfrak{g}$ on $C^\infty(\Gamma\diagdown G) \subset L^2(\Gamma\diagdown G)$; i.e., for $X \in \mathfrak{g}$, $f \in C^\infty(\Gamma\diagdown G)$,

$$(\rho_*(X) f)(x) = \left.\frac{d}{dt}\right|_0 f(x \exp(tX)).$$

Because on smooth functions $\rho_*(X) f = Xf$, we may rewrite the Laplacian as

$$\Delta = -\sum (\rho_* X_i)^2.$$

By expressing the Laplace-Beltrami operator in terms of the representation ρ, we see that irreducible subspaces of the representation are also invariant subspaces of the Laplacian. By restricting Δ to an irreducible subspace of $L^2(\Gamma\diagdown G)$, Gordon, Wilson, and Pesce ([20], [26]) have been able in the two-step nilpotent case to explicitly solve for its eigenvalues and eigenfunctions. The Laplace spectrum of $\Gamma\diagdown G$ is then the union over all irreducible subspaces of the spectrum of the restricted Laplacian. The multiplicity of an eigenvalue is the sum over the irreducible subspaces of $L^2(\Gamma\diagdown G)$ of the eigenvalue's multiplicity in the irreducible subspace times the multiplicity of the irreducible subspace in $L^2(\Gamma\diagdown G)$. The key ingredient that distinguishes the nilpotent case in general, and the two-step nilpotent case in particular, is that occurrence conditions, eigenvalues, eigenfunctions, and multiplicities can be explicitly expressed in terms of $\log\Gamma$ and $(\mathfrak{g}, \langle\ ,\ \rangle)$ using Kirillov theory. For more details, see [19].

Kirillov ([22], [23]) proved that equivalence classes of irreducible unitary representations of nilpotent Lie groups G are in 1-1 correspondence with the orbits of the coadjoint action of G on $\mathfrak{g}^*$. The coadjoint action is defined by, for $x \in G$, $\alpha \in \mathfrak{g}^*$,

$$x \cdot \alpha = \alpha \circ \mathrm{Ad}\left(x^{-1}\right).$$

Given a fixed representative $\alpha \in \mathfrak{g}^*$ corresponding to a coadjoint orbit, let π_α denote the associated irreducible unitary representation of G with representation space W_α. The possible dimensions of W_α are either 1 (characters) or infinite. L. F. Richardson ([28]) computed the decomposition of ρ into irreducibles.

Notation: Given $\alpha \in \mathfrak{g}^*$, let $B_\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be defined by

$$B_\alpha(X, Y) = \alpha([X, Y]).$$

Let $\mathfrak{g}_\alpha = \ker(B_\alpha) = \{X \in \mathfrak{g} : B_\alpha(X, Y) = 0 \text{ for all } Y \in \mathfrak{g}\}$, let $\overline{B_\alpha}$ be the nondegenerate skew-symmetric bilinear form induced by B_α on $\mathfrak{g}\diagup\mathfrak{g}_\alpha$, and denote by $\pm i\ d_1, ..., \pm i\ d_r$ the eigenvalues of $\overline{B_\alpha}$. Note $\log\Gamma$ generates a lattice $\mathcal{L}$ in $\mathfrak{g}$. Let $\mathcal{A}_\alpha = \mathcal{L}\diagup(\mathcal{L} \cap \mathfrak{g}_\alpha)$. Let

$$\Delta_\alpha = \left.\Delta\right|_{W_\alpha}.$$

In the two-step nilpotent case, H. Pesce explicitly calculated the spectrum of the restricted Laplace-Beltrami operator Δ_α as follows.

Proposition 6. *(Pesce) We continue the notation above.*

(1) π_α *occurs in the representation* $L^2(\Gamma\diagdown G)$ *iff*

$$\alpha(\log\Gamma \cap \mathfrak{g}_\alpha) \subset \mathbb{Z}.$$

(2) *If π_α occurs and $\alpha(\mathfrak{z}) = \{0\}$, then π_α is one-dimensional and occurs with multiplicity $m_a = 1$. The Laplace spectrum associated to this irreducible subspace is*
$$spec(\Delta_\alpha) = \left\{4\pi^2 \|\alpha\|^2\right\}.$$
(3) *If π_α occurs and $\alpha(\mathfrak{z}) \neq \{0\}$, then π_α is infinite-dimensional and occurs with multiplicity*
$$m_\alpha = \sqrt{\det\left(\overline{B_\alpha}\right)},$$
where the determinant is computed with respect to (any) lattice basis of $\mathcal{A}_\alpha \subset \mathfrak{g}\diagup\mathfrak{g}_\alpha$. The Laplace spectrum associated to this irreducible subspace is
$$\operatorname{spec}(\Delta_\alpha) = \{\mu(\alpha, p) : p \in (\mathbb{Z}_{\geq 0})^r\},$$
where
$$\mu(\alpha, p) = 4\pi^2 \sum \alpha(Z_i)^2 + 2\pi \sum (2p_k + 1)\, d_k,$$
with $\{Z_1, ..., Z_l\}$ an orthonormal basis of $\mathfrak{g}_\alpha$. The multiplicity of μ in $spec(\Delta_\alpha)$ is the number of $p \in (\mathbb{Z}_{\geq 0})^r$ satisfying $\mu(\alpha, p) = \mu$.

Remark 7. *In other words, the multiplicity of an eigenvalue λ is the sum of the multiplicity of λ as an eigenvalue in each Δ_α times the multiplicity of π_a in the representation $L^2(\Gamma\diagdown G)$.*

3.2. **The Dirac operator on two-step nilmanifolds.** As we intend to calculate the eta invariant of the spin^c Dirac operator, we now extend Pesce's results to the Dirac setting. Recall that G is a simply connected n-dimensional two-step nilpotent Lie group with Lie algebra $\mathfrak{g}$ and Γ is a cocompact, discrete subgroup of G. We fix an inner product on $\mathfrak{g}$, which corresponds to a left-invariant metric on G, which descends to a Riemannian metric on $\Gamma\diagdown G$.

Let Σ_n be a standard irreducible spinor representation (see [6, Section 3.2]), also considered as a trivial bundle over G. A spin structure and the corresponding spinor bundle Σ_ε over $\Gamma\diagdown G$ are determined by Σ_n and a homomorphism $\varepsilon : \Gamma \to \{\pm 1\}$ (see [6, Prop 3.34, p. 114]). We have
$$L^2(\Gamma\diagdown G, \Sigma_\varepsilon) \cong L^2_\varepsilon(\Gamma\diagdown G) \otimes_{\mathbb{C}} \Sigma_n, \tag{12}$$
where $L^2_\varepsilon(\Gamma\diagdown G)$ is defined by
$$L^2_\varepsilon(\Gamma\diagdown G) = \left\{f \in L^2_{\text{loc}}(G) : f(\gamma x) = \varepsilon(\gamma) f(x) \text{ for all } \gamma \in \Gamma, x \in G\right\}. \tag{13}$$
The isomorphism from $L^2_\varepsilon(\Gamma\diagdown G) \otimes_{\mathbb{C}} \Sigma_n$ to $L^2(\Gamma\diagdown G, \Sigma_\varepsilon)$ is $f \otimes s \mapsto fs$, where Σ_n is identified with the constant sections $G \to G \times \Sigma_n$. Clifford multiplication by elements of $T(\Gamma\diagdown G) \cong \Gamma\diagdown G \times \mathfrak{g}$ is given by the standard Clifford action $\diamond$ of $\mathbb{C}\mathrm{l}(\mathfrak{g})$ on Σ_n. That is, $\xi \in \mathfrak{g}$ acts on $L^2_\varepsilon(\Gamma\diagdown G) \otimes_{\mathbb{C}} \Sigma_n$ by
$$\xi \diamond (fs) = f(\xi \diamond s).$$
By construction, $(\xi\diamond)$ is a constant matrix on $\Gamma\diagdown G$ for every left-invariant vector field ξ.

Note that the (Clifford) connection on any spinor bundle is given by
$$\nabla^\Sigma_{E_i} = \partial_{E_i} + \frac{1}{4} \sum_{j,k} \Gamma^k_{ij} (E_j\diamond)(E_k\diamond) \tag{14}$$
according to the Ammann-Bär formula [1, formula 1.1], where $\{E_j\}$ is a left-invariant orthonormal basis of the tangent space, Γ^k_{ij} are the Christoffel symbols associated to the metric and frame, and ∂_{E_i} is a directional derivative. In our case, we use the left-invariant metric

on $\mathfrak{g}$, yielding a (no longer left-invariant) metric on $\Gamma\diagdown G$. Then the Dirac operator D on $\Gamma\diagdown G$ acts on $L^2_\varepsilon(\Gamma\diagdown G)\otimes\Sigma_n$ by

$$\begin{aligned} D &= \sum (E_i\diamond)\nabla^\Sigma_{E_i} \\ &= \sum_i (E_i\diamond)\partial_{E_i}+\frac{1}{4}\sum_{i,j,k}\Gamma^k_{ij}(E_i\diamond E_j\diamond E_k\diamond) \end{aligned}$$

If ρ_ε denotes right multiplication acting on $L^2_\varepsilon(\Gamma\diagdown G)$, we have

$$\begin{aligned} \partial_{E_i} &= \left.\frac{d}{dt}\right|_0 \rho_\varepsilon(\exp(tE_i)) \\ &= \rho_{\varepsilon *}(E_i). \end{aligned}$$

Note that ρ_ε is the induced representation of $\varepsilon:\Gamma\to\{\pm1\}$ to G. The Christoffel symbols are defined by

$$\nabla_{E_i}E_j=\sum\Gamma^k_{ij}E_k,$$

and the Koszul formula gives

$$2\Gamma^k_{ij}=-\langle E_i,[E_j,E_k]\rangle+\langle E_j,[E_k,E_i]\rangle+\langle E_k,[E_i,E_j]\rangle.$$

At this point, the formulas given above are completely general for any Lie group G with a left-invariant metric.

We now assume G is 2-step nilpotent, so that $\langle E_i,[E_j,E_k]\rangle=0$ unless E_i is in the center of $\mathfrak{g}$. If $\mathfrak{g}=\mathfrak{z}\oplus\mathfrak{v}$ with $\mathfrak{z}$ the center and $\mathfrak{v}=\mathfrak{z}^\perp$, its orthogonal complement, then the inner product on $\mathfrak{g}$ is determined by and determines the map $j:\mathfrak{z}\to\mathfrak{so}(\mathfrak{v})$ defined as

$$\langle j(Z)X,A\rangle=\langle Z,[X,A]\rangle \tag{15}$$

for all $Z\in\mathfrak{z}$ and all $X,A\in\mathfrak{v}$. See, for example, [14].

Let k_0 be the dimension of the center and k_0+m_0 the dimension of $\mathfrak{g}$, and we choose the orthonormal basis $\{Z_1,...,Z_{k_0},X_1,...,X_{m_0}\}$ so that $\{Z_i\}$ is an orthonormal basis of $\mathfrak{z}$ and $\{X_i\}$ is an orthonormal basis of $\mathfrak{v}$. Then one easily verifies that

$$\nabla_{Z_i}X_k=\nabla_{X_k}Z_i=-\frac{1}{2}j(Z_i)X_k,\ \nabla_{X_i}X_k=\frac{1}{2}[X_i,X_k],\ \nabla_{Z_i}Z_k=0.$$

We label $E_1=Z_1,...,E_{k_0}=Z_{k_0},E_{k_0+1}=X_1,...,E_{k_0+m_0}=X_{m_0}$. The Christoffel symbols satisfy $\Gamma^r_{pq}=0$ if at least two of p,q,r are $\le k_0$ or if $p,q,r>k_0$. If $a\le k_0$, $b,q>k_0$,

$$\begin{aligned} 2\Gamma^a_{bq} &= -2\Gamma^a_{qb}=2\Gamma^b_{aq}=2\Gamma^b_{qa}=-2\Gamma^q_{ab}=-2\Gamma^q_{ba} \\ &= \langle Z_a,[X_{b-k_0},X_{q-k_0}]\rangle=\langle j(Z_a)X_{b-k_0},X_{q-k_0}\rangle. \end{aligned}$$

Letting $\partial_{E_i} = \partial_i$, $C_i = (E_i \diamond)$, $C_{abq} = (E_a \diamond E_b \diamond E_q \diamond)$, etc., the Dirac operator is

$$\begin{aligned} D &= \sum_i \partial_i C_i + \frac{1}{4} \sum_{i,j,k} \Gamma_{ij}^k C_{ijk} \\ &= \sum_i \partial_i C_i + \frac{1}{4} \sum_{a \le k_0;\ b,q > k_0} \Gamma_{bq}^a C_{bqa} + \Gamma_{aq}^b C_{aqb} + \Gamma_{ba}^q C_{baq} \\ &= \sum_i \partial_i C_i + \frac{1}{4} \sum_{a \le k_0;\ q > b > k_0} \Gamma_{bq}^a C_{bqa} + \Gamma_{qb}^a C_{qba} + \Gamma_{aq}^b C_{aqb} + \Gamma_{ab}^q C_{abq} + \Gamma_{ba}^q C_{baq} + \Gamma_{qa}^b C_{qab} \\ &= \sum_i \partial_i C_i + \frac{1}{2} \sum_{a \le k_0;\ q > b > k_0} \Gamma_{bq}^a C_{abq} \\ &= \sum_i \partial_i C_i + \frac{1}{4} \sum_{a \le k_0;\ q > b > k_0} \langle Z_a, [X_{b-k_0}, X_{q-k_0}] \rangle \left(Z_a \diamond X_{b-k_0} \diamond X_{q-k_0} \diamond \right), \end{aligned}$$

so

$$\begin{aligned} D &= \sum_i (E_i \diamond) \, \partial_{E_i} + \frac{1}{4} \sum_{a \le k_0;\ b < i \le m_0} \langle Z_a, [X_b, X_i] \rangle \left(Z_a \diamond X_b \diamond X_i \diamond \right) \\ &= \sum_i (E_i \diamond) \, \rho_{\varepsilon *} (E_i) + \frac{1}{4} \sum_{a \le k_0;\ b < i \le m_0} \langle Z_a, [X_b, X_i] \rangle \left(Z_a \diamond X_b \diamond X_i \diamond \right). \qquad (16) \end{aligned}$$

The formula above works for any two-step nilmanifold.

Remark 8. *In the three-dimensional Heisenberg case, for some constant $A > 0$, we let $\left\{ X_1 = \frac{1}{\sqrt{A}} X, X_2 = \frac{1}{\sqrt{A}} Y, Z \right\}$ be an orthonormal frame with $[X, Y] = Z$. We choose a basis of $\Sigma_3 \cong \mathbb{C}^2$ so that*

$$(Z \diamond) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ (X_1 \diamond) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ (X_2 \diamond) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} \langle Z, [X_1, X_2] \rangle &= \frac{1}{A}, \\ (Z \diamond X_1 \diamond X_2 \diamond) &= -\mathbf{1}, \end{aligned}$$

so the equation above becomes

$$D = \sum_{i=1}^{3} (E_i \diamond) \, \partial_{e_i} - \frac{1}{4A},$$

as seen in [1, Equation 3.2]*, with $d^2 T = \frac{1}{A}$ in their notation.*

3.3. **Analogue of Pesce's theorem for spinors.** In this section, we decompose $L^2_\varepsilon (\Gamma \diagdown G)$ as a direct sum of irreducible representations. Let $\alpha \in \mathfrak{g}^*$. Recall $B_\alpha (X, Y) := \alpha ([X, Y])$, $\mathfrak{g}_\alpha = \ker B_\alpha$, so that $\alpha ([\mathfrak{g}_\alpha, \mathfrak{g}]) = 0$. Let $\mathfrak{g}^\alpha$ be a maximal polarizer of α, meaning that it is a subalgebra of $\mathfrak{g}$ such that $\alpha ([\mathfrak{g}^\alpha, \mathfrak{g}^\alpha]) = 0$ and there does not exist a subalgebra $\mathfrak{g}'$ with the same property such that $\mathfrak{g}^\alpha \subsetneq \mathfrak{g}' \subseteq \mathfrak{g}$. Note that for every $\alpha \in \mathfrak{g}^*$ and every choice of $\mathfrak{g}^\alpha$,

$$\mathfrak{g}_\alpha \subset \mathfrak{g}^\alpha.$$

Given $\mathfrak{g}^\alpha$, let $G^\alpha = \exp (\mathfrak{g}^\alpha)$.

Lemma 9. *(Lemma 4 from* [26, Appendix A]*) Let $\alpha \in \mathfrak{g}^*$, $\alpha(\mathfrak{z}) \neq 0$ and $B_\alpha(X,Y) = \alpha([X,Y]) \in \mathbb{Z}$ for all $X, Y \in \log\Gamma$. Then there exists a basis $\{U_1,...,U_m,V_1,...,V_m,W_1,...,W_k\}$ of $\mathfrak{g}$ formed of elements of $\log\Gamma$, and there exist integers $r_1,...,r_k$ such that*

(1) *We have*

$$\begin{aligned} B_\alpha(U_i,V_i) &= \alpha([U_i,V_i]) = r_i, \\ B_\alpha(U_i,V_j) &= 0 \text{ if } i \neq j, \text{ and} \\ B_\alpha(U_i,U_j) &= B_\alpha(V_i,V_j) = 0 \text{ for all } i,j. \end{aligned}$$

(2) *$\{W_1,...,W_k\}$ is a basis of $\mathfrak{g}_\alpha$, $\{W_1,...,W_{k_0}\}$ is a basis of $\mathfrak{z}$, $k_0 \leq k$,*

(3) *$\mathfrak{z} \cap \log\Gamma = \mathrm{span}_{\mathbb{Z}}\{W_1,...,W_{k_0}\}$.*

As before, $\log\Gamma$ generates a lattice $\mathcal{L}$ in $\mathfrak{g}$. Let $\mathcal{A}_\alpha = \mathcal{L}\big/(\mathcal{L}\cap\mathfrak{g}_\alpha)$. When π_α occurs, this will be a lattice in $\mathfrak{g}\big/\mathfrak{g}_\alpha$.

Proposition 10. *(Version of Pesce Occurrence Condition (*[26, Proposition 9 of Appendix A]*) for Dirac spinors) The representation π_α appears in $L^2_\varepsilon(\Gamma\diagdown G)$ if and only if*

$$\alpha(\log\gamma) \in \mathbb{Z} + \frac{1-\varepsilon(\gamma)}{4} \tag{17}$$

for all $\gamma \in \Gamma\cap G_\alpha$. In this case, the multiplicity of π_α is $m_\alpha = 1$ if $\alpha(\mathfrak{z}) = \{0\}$, and otherwise

$$m_\alpha = \sqrt{\det\left(\overline{B_\alpha}\right)},$$

where the determinant is computed with respect to (any) lattice basis of $\mathcal{A}_\alpha \subset \mathfrak{g}\big/\mathfrak{g}_\alpha$.

Proof. Items 4 through 8 in [26, Appendix A] apply in this situation.

If $\alpha(\mathfrak{z}) = 0$, then $\mathfrak{g} = \mathfrak{g}_\alpha = \mathfrak{g}^\alpha$. Then condition (17) is equivalent to Theorem 20. In addition, using Theorem 22,

$$\begin{aligned} m\left(\pi_\alpha, L^2_\varepsilon(\Gamma\diagdown G)\right) &= \#\left((G^\alpha\diagdown G)_\varepsilon\big/\Gamma\right) \\ &= \#\left((G\diagdown G)_\varepsilon\big/\Gamma\right) = 1. \end{aligned}$$

For the remainder of the proof, we assume $\alpha(\mathfrak{z}) \neq 0$. First, we assume π_α appears in $L^2_\varepsilon(\Gamma\diagdown G)$. Then, by Theorem 20, there exists α' in the coadjoint orbit of α such that $\left(\overline{\alpha'}, G^{\alpha'}\right)$ is an ε-integral point, where $\overline{\alpha'} = \overline{\alpha}\circ I_x$ (I_x = conjugation by x), $\alpha' = \alpha\circ Ad(x)$ and $G^{\alpha'} = I_{x^{-1}}(G^\alpha)$ such that

$$\alpha'(\log\gamma) \in \begin{cases} \mathbb{Z} & \text{if } \varepsilon(\gamma) = 1 \\ \frac{1}{2} + \mathbb{Z} & \text{if } \varepsilon(\gamma) = -1 \end{cases}$$

for all $\gamma \in \Gamma\cap G^{\alpha'}$. In the two-step case, $G^{\alpha'} = G^\alpha$ since $I_x(y)\,y^{-1} \in Z(G)$ for all $x, y \in G$, and $Z(G) \subseteq G_\alpha \subseteq G^{\alpha'}$. Also in the two-step case, if α, α' lie in the same coadjoint orbit, then there exists $X \in \mathfrak{g}$ such that $\alpha' = \alpha\circ(I + \mathrm{ad}(X))$. Thus

$$\alpha(\log\gamma + \mathrm{ad}(X)\log\gamma) \in \begin{cases} \mathbb{Z} & \text{if } \varepsilon(\gamma) = 1 \\ \frac{1}{2} + \mathbb{Z} & \text{if } \varepsilon(\gamma) = -1 \end{cases}$$

for all $\gamma \in \Gamma\cap G^\alpha$. This implies the same condition is met for all $\gamma \in \Gamma\cap G_\alpha$, in which case $\alpha(\mathrm{ad}(X)\log\gamma) = \alpha([X,\log\gamma]) = 0$, by definition of G_α.

On the other hand, suppose

$$\alpha(\log\gamma) \in \mathbb{Z} + \frac{1-\varepsilon(\gamma)}{4}$$

for all $\gamma \in \Gamma \cap G_\alpha$. Note that if $X, Y \in \log \Gamma$, then $[X, Y] \in \log \Gamma$ since $[\exp X, \exp Y] = \exp([X, Y])$ (since G is two-step). Therefore, $\alpha([X, Y]) \in \mathbb{Z}$, since $\varepsilon([\exp X, \exp Y]) = 1$. We can then use Lemma 9 to construct a basis $\{U_1, ..., U_m, V_1, ..., V_m, W_1, ..., W_k\} \subset \log \Gamma$ of $\mathfrak{g}$ and integers $r_1, ..., r_m$ such that $\alpha([U_j, V_j]) = r_j$. Set $\mathfrak{h} = \operatorname{span}_{\mathbb{R}} \{V_1, ..., V_m, W_1, ..., W_k\}$. Then $\mathfrak{h}$ is a rational ideal of $\mathfrak{g}$, since $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{z} \subseteq \mathfrak{h}$ (two-step condition), and $\mathfrak{h}$ is a polarizer of α. Set $H = \exp(\mathfrak{h})$, which is a normal subgroup of G. Note that

$$H = \left\{ \prod_{i=1}^{m} \exp(y_i V_i) \prod_{j=1}^{k} \exp(z_j W_j) : y_i, z_j \in \mathbb{R} \right\}.$$

Define $\overline{\alpha}(\exp(X)) = \exp(2\pi i \alpha(X))$ for all $X \in \mathfrak{h}$. By Theorem 20, to prove that π_α occurs, we need only construct an ε-integral point in the G-orbit of $(\overline{\alpha}, H)$. For $x \in G$, define x_i, y_i, z_j by the formula

$$x = \prod_{i'=1}^{m} \exp(x_{i'} U_{i'}) \prod_{i=1}^{m} \exp(y_i V_i) \prod_{j=1}^{k} \exp(z_j W_j),$$

and define p_i, q_i, η_j by

$$\alpha\left(\sum u_i U_i + v_i V_i + \sum w_j W_j\right) = \sum (p_i u_i + q_i v_i) + \sum \eta_j w_j,$$

for all $u_i, v_i, w_j \in \mathbb{R}$, $1 \le i \le m, 1 \le j \le k$. By [26, Lemma 7, Appendix A], [26, Theorem 8, Appendix A],

$$H \cap \Gamma = \left\{ \prod_{i=1}^{m} \exp(t_i V_i) \prod_{j=1}^{k} \exp(s_j W_j) : t_i, s_j \in \mathbb{Z} \right\}.$$

We need to show that there exists $x \in G$ such that $(\overline{\alpha} \circ I_x)(\gamma) = \varepsilon(\gamma)$ for all $\gamma \in H \cap \Gamma$. First note that

$$\begin{aligned} (\overline{\alpha} \circ I_x)(\exp(W_j)) &= \overline{\alpha}(\exp(W_j + [\log(x), W_j])) \\ &= \exp(2\pi i \alpha(W_j + [\log(x), W_j])) \\ &= \exp(2\pi i \alpha(W_j)) \text{ since } W_j \in \mathfrak{g}_\alpha \\ &= \overline{\alpha}(\exp(W_j)) = \varepsilon(\exp(W_j)) \end{aligned}$$

since $W_j \in (\log \Gamma) \cap \mathfrak{g}_\alpha$. Next,

$$\begin{aligned} (\overline{\alpha} \circ I_x)(\exp(V_j)) &= \overline{\alpha}(\exp(V_j + [\log(x), V_j])) \\ &= \exp(2\pi i \alpha(V_j + [\log(x), V_j])) \\ &= \exp(2\pi i \alpha(V_j + x_j [U_j, V_j])) \\ &= \exp(2\pi i (q_j + x_j r_j)). \end{aligned} \tag{18}$$

By setting $x_j = -\frac{q_j}{r_j}$ or $-\frac{q_j + \frac{1}{2}}{r_j}$ depending on whether $\varepsilon(\exp(V_j)) = \pm 1$, we conclude.

$$(\overline{\alpha} \circ I_x)(\exp(V_j)) = \varepsilon(\exp(V_j)).$$

We have shown that for all $\gamma \in H \cap \Gamma$ there exists $x \in G$ such that $(\overline{\alpha} \circ I_x)(\gamma) = \varepsilon(\gamma)$.

We now calculate the multiplicity with which π_α appears. In fact, for $x \in G$, the calculations above show that for all $X \in \mathfrak{h}$, $(\overline{\alpha} \circ I_x)(\exp(X))$ depends only on the x_i and not on y_i or z_j, $1 \le i \le m, 1 \le j \le k$. Thus from (18) the orbit of $(\overline{\alpha}, H)$ is the set of characters of H

$$\{(\chi_{q'}, H) : q' \in \mathbb{R}^m\},$$

where where after a bit of calculation identical to [26, p.453, lines -8 through -5]

$$\chi_{q'}\left(\prod_{i=1}^{m}\exp\left(t_iV_i\right)\prod_{j=1}^{k}\exp\left(s_jW_j\right)\right)=\exp\left(2\pi i\left(\sum_{i=1}^{m}q_i't_i+\sum_{j=1}^{k}\eta_js_j\right)\right).$$

Then $(\chi_{q'},H)$ is an ε-integer point if and only if

$$\begin{aligned} q_i' &\in \mathbb{Z}\text{ whenever }\varepsilon\left(\exp\left(V_i\right)\right)=1,\\ q_i' &\in \frac{1}{2}+\mathbb{Z}\text{ whenever }\varepsilon\left(\exp\left(V_i\right)\right)=-1. \end{aligned}$$

Note also from (18) that two ε-integer points $(\chi_{q'},H)$ and $(\chi_{q''},H)$ are in the same Γ-orbit if and only if $q_i'-q_i''\in r_i\mathbb{Z}$, $i=1,...,m$. So the number m_α of Γ-orbits in the ε-integer points is $r_1r_2...r_m$. Next, it is clear that the images of $U_1,...,U_m,V_1,...,V_m$ form a basis of $\mathcal{A}_\alpha$. So

$$\det\left(\overline{B_\alpha}\right)=\det\left(B_\alpha\left(U_i,V_j\right)\right)^2=\left(r_1r_2...r_m\right)^2=m_\alpha^2.$$

□ □

4. Decomposition of the Dirac operator on two-step nilmanifolds

We continue with the notation of the previous section; recall that k_0 is the dimension of the center $\mathfrak{z}$ and $n=k_0+m_0$ is the dimension of $\mathfrak{g}=\mathfrak{z}\oplus\mathfrak{v}$, and we will choose the orthonormal basis $\{E_1,...,E_n\}=\{Z_1,...,Z_{k_0},X_1,...,X_{m_0}\}$ so that $\{Z_j\}$ is an orthonormal basis of $\mathfrak{z}$ and $\{X_j\}$ is an orthonormal basis of $\mathfrak{v}$. From formula (16), the Dirac operator is

$$D=\sum_{i=1}^{n}\left(E_i\diamond\right)\rho_{\varepsilon *}\left(E_i\right)+\frac{1}{4}\sum_{a\leq k_0;\ b<i\leq m_0}\left\langle Z_a,\left[X_b,X_i\right]\right\rangle\left(Z_a\diamond X_b\diamond X_i\diamond\right),$$

acting on

$$\mathcal{H}=L^2\left(\Gamma\diagdown G,G\times_\varepsilon\mathbb{C}^k\right)\cong L_\varepsilon^2\left(\Gamma\diagdown G\right)\otimes\Sigma_n, \tag{19}$$

which we decompose using Kirillov theory.

Choose an element $\alpha\in\mathfrak{g}^*$. Our strategy is as follows. We first construct a subspace $\mathcal{H}_\alpha$ of $L^2\left(\Gamma\diagdown G,G\times_\varepsilon\mathbb{C}^k\right)$ that is invariant with respect to ρ_ε and invariant by D. Once we have done this, by Kirillov theory, let $\overline{\mathcal{H}_\alpha}$ be the irreducible ρ_ε-subspace of $L_\varepsilon^2\left(\Gamma\diagdown G\right)$ corresponding to the coadjoint orbit of α, and let

$$\mathcal{H}_\alpha\cong\overline{\mathcal{H}_\alpha}\otimes\Sigma_n$$

through the isomorphism above. While $\overline{\mathcal{H}_\alpha}$ is ρ_ε-irreducible, $\mathcal{H}_\alpha$ is not for $n\geq 2$. We express D acting on $\mathcal{H}_\alpha$, and because of the two-step structure, we are able to solve explicitly the partial differential equation for eigenvalues via Hermite functions.

Since $\sum_{i=1}^{n}\rho_{\varepsilon *}\left(E_i\right)\left(E_i\diamond\right)$ is independent of the choice of basis $\{E_1,...,E_n\}$, the second term is similarly independent of choices and independent of the representation ρ_ε. Define

$$\begin{aligned} D_{\rho_\varepsilon} &= \sum_{i=1}^{n}\left(E_i\diamond\right)\rho_{\varepsilon *}\left(E_i\right)\\ M &= \frac{1}{4}\sum_{a\leq k_0;\ b<i\leq m_0}\left\langle Z_a,\left[X_b,X_i\right]\right\rangle\left(Z_a\diamond X_b\diamond X_i\diamond\right), \end{aligned} \tag{20}$$

so that $D=D_{\rho_\varepsilon}+M$ with M a hermitian linear transformation independent of invariant subspace. Note that ρ_ε and $(Y\diamond)$ commute if $(Y\diamond)$ is a constant transformation — that is,

if Y is left-invariant. Thus M commutes with ρ_ε because each $\langle Z_a, [X_b, X_i]\rangle$ is constant on $\Gamma\diagdown G$.

As before, we define the symplectic form on $\mathfrak{g}$ by $B_\alpha(U,V) := \alpha([U,V])$, and let $\mathfrak{g}_\alpha = \ker B_\alpha = \{U \in \mathfrak{g} : B_\alpha(U,\cdot) = 0\}$, $k_\alpha = \dim \mathfrak{g}_\alpha$. We have two cases.

4.1. **Finite-dimensional $\overline{\mathcal{H}_\alpha}$-irreducible subspaces: $k_\alpha = n$, i.e. $\alpha([\mathfrak{g},\mathfrak{g}]) = 0$.** In this case, $\mathfrak{g}_\alpha = \mathfrak{g}$, and $\mathfrak{g}^\alpha = \mathfrak{g}$ is a maximal polarizer of α. Then $G^\alpha = \exp(\mathfrak{g}^\alpha) = G$. Define

$$\begin{aligned} \mathcal{H}_\alpha &= \{\sigma \in \mathcal{H} : \sigma(hx) = \overline{\alpha}(h)\sigma(x) \text{ for all } h \in G^\alpha, x \in G\} \\ &= \{\sigma \in \mathcal{H} : \sigma(hx) = \overline{\alpha}(h)\sigma(x) \text{ for all } h \in G, x \in G\} \\ &= \overline{\alpha}(\cdot) \otimes \Sigma_n, \end{aligned}$$

where

$$\overline{\alpha}(h) = e^{2\pi i\alpha(\log h)}.$$

For $\sigma \in \mathcal{H}_\alpha$, we have, since $\alpha([\mathfrak{g},\mathfrak{g}]) = 0$, for $p \in \Gamma\diagdown G$,

$$\begin{aligned} \rho_{\varepsilon *}(U)\sigma(p) &= \left.\frac{d}{dt}\right|_0 \sigma(p\exp(tU)\mathbf{1}) = \left.\frac{d}{dt}\right|_0 e^{2\pi i\alpha\log(p\exp(tU))}\sigma(\mathbf{1}) \\ &= \left.\frac{d}{dt}\right|_0 e^{2\pi i\alpha\left(\log(p)+tU+\frac{1}{2}[\log(p),tU]\right)}\sigma(\mathbf{1}) \\ &= \left.\frac{d}{dt}\right|_0 e^{2\pi i\alpha(\log(p)+tU)}\sigma(\mathbf{1}). \end{aligned}$$

We have $\rho_{\varepsilon *}(Z_a)\sigma = 0$, and $\rho_{\varepsilon *}(X_i)\sigma = \frac{\partial}{\partial x_i}\sigma = 2\pi i\alpha(X_i)\sigma$. Thus,

$$\begin{aligned} D|_{\mathcal{H}_\alpha} &= \sum_{i=1}^{m_0} 2\pi i\alpha(X_i)(X_i\diamond) + \sum_{Z_j\notin[\mathfrak{g},\mathfrak{g}]} 2\pi i\alpha(Z_j)(Z_j\diamond) \\ &\quad + \frac{1}{4}\sum_{a\le k_0;\ b<i\le m_0} \langle Z_a, [X_b, X_i]\rangle (Z_a \diamond X_b \diamond X_i\diamond), \end{aligned} \tag{21}$$

which is a constant matrix. The eigenvalues of $D|_{\mathcal{H}_\alpha}$ are then the eigenvalues of this Hermitian matrix.

4.2. **Infinite-dimensional $\overline{\mathcal{H}_\alpha}$-irreducible subspaces: $k_\alpha < n$, so that $\alpha([\mathfrak{g},\mathfrak{g}])$ is not identically zero.** Choose a new orthonormal basis of $\mathfrak{g}$:

$$\{W_1, ..., W_{k_\alpha}, U_1, ..., U_m, V_1, ..., V_m\},$$

where $n = k_\alpha + 2m$, $\{W_j\}$ is a basis of $\mathfrak{g}_\alpha$ with $W_1 = Z_1, ..., W_{k_0} = Z_{k_0} \in \mathfrak{z}$, $W_{k_0+1}, ..., W_{k_\alpha} \in \mathfrak{g}_\alpha \cap \mathfrak{z}^\perp$.

$$\begin{aligned} B_\alpha(U_i, V_i) &= \alpha([U_i, V_i]) = d_i > 0, 0 < d_1 \le d_2 \le \cdots \le d_m, \\ B_\alpha(U_i, V_j) &= 0 \text{ if } i \ne j \\ B_\alpha(U_i, U_j) &= B_\alpha(V_i, V_j) = 0 \text{ for all } i, j. \end{aligned}$$

Note the similarity with Lemma 9, but we have replaced some of the V_j with their negatives in order to make d_j positive. We may assume $n - k_\alpha$ is even, since the restriction of B_α to $\mathfrak{g}_\alpha^\perp$ is a symplectic form. Then the polarizing subalgebra $\mathfrak{g}^\alpha$ (meaning that $\mathfrak{g}^\alpha$ is a subalgebra of $\mathfrak{g}$ such that $\alpha([\mathfrak{g}^\alpha, \mathfrak{g}^\alpha]) = 0$ and is maximal with respect to inclusion) will be chosen to be

$$\mathfrak{g}^\alpha = \operatorname{span}\{V_1, ..., V_m, W_1, ..., W_{k_\alpha}\},$$

and again $G^{\alpha} := \exp(\mathfrak{g}^{\alpha})$. We have, with $\overline{\alpha}(h) = \exp(2\pi i\alpha(\log h))$,

$$\mathcal{H}_{\alpha} = \{\sigma \in \mathcal{H} : \sigma(hx) = \overline{\alpha}(h)\,\sigma(x) \text{ for all } h \in G^{\alpha}, x \in G\}.$$

Let $\overline{\mathcal{H}_{\alpha}}$ be the ρ_{ε}-irreducible subspace of $L^2_{\varepsilon}(\Gamma\diagdown G)$ such that

$$\mathcal{H}_{\alpha} \cong \overline{\mathcal{H}_{\alpha}} \otimes \Sigma_n$$

through the isomorphism (12). Let $\beta : \overline{\mathcal{H}_{\alpha}} \to L^2_{\mathbb{C}}(\mathbb{R}^m)$ be the unitary isomorphism defined by $\beta(F)(t) = F(\exp(t_1U_1)\ldots\exp(t_mU_m))$. Note that the map

$$t \in \mathbb{R}^k \mapsto G^{\alpha}\exp(t_1U_1)\ldots\exp(t_mU_m) \in G^{\alpha}\diagdown G$$

pushes the Euclidean metric onto a right G-invariant metric on $G^{\alpha}\diagdown G$. Note that $\overline{\mathcal{H}_{\alpha}} = \beta^{-1}(L^2_{\mathbb{C}}(\mathbb{R}^m))$, and for $x = h\exp(t_1U_1)\ldots\exp(t_mU_m)$ an arbitrary element of G with $h \in G^{\alpha}$, and $f \in L^2_{\mathbb{C}}(\mathbb{R}^m)$,

$$\begin{aligned}\left(\beta^{-1}f\right)(x) &= \left(\beta^{-1}f\right)(h\exp(t_1U_1)\ldots\exp(t_mU_m)) \\ &= \overline{\alpha}(h)\,f(t_1,\ldots,t_m).\end{aligned}$$

Here π_{α} is the representation of G on $\overline{\mathcal{H}_{\alpha}}$ induced from the character $\overline{\alpha}$ of G^{α}; we have for $f \in \overline{\mathcal{H}_{\alpha}}$,

$$(\pi_{\alpha}(x)\,f)(y) = (\rho_{\varepsilon x}f)(y) = f(yx).$$

We define the representation π'_{α} of G on $L^2_{\mathbb{C}}(\mathbb{R}^m)$ by

$$\pi'_{\alpha}(x) = \beta \circ \pi_{\alpha}(x) \circ \beta^{-1}$$

for all $x \in G$.

For any $x, y \in G$, let $[x, y] = xyx^{-1}y^{-1}$. To compute the action of G on $L^2_{\mathbb{C}}(\mathbb{R}^m)$, recall that since G is 2-step (see [26, Section 3]), for any $h_0 \in G^{\alpha}$,

- $\prod\limits_{j=1}^{m}\exp(t_jU_j)\,h_0 = \left[\prod\limits_{j=1}^{m}\exp(t_jU_j), h_0\right] h_0 \prod\limits_{j=1}^{m}\exp(t_jU_j)$
- $\prod\limits_{l=1}^{m}\exp(t_lU_l)\prod\limits_{j=1}^{m}\exp(s_jU_j) = \prod\limits_{j=1}^{m}\exp((t_j+s_j)\,U_j)\exp\left(-\sum_{1\le j<l\le m} t_ls_j\,[U_j, U_l]\right)$
- $\left[\prod\limits_{j=1}^{m}\exp(t_jU_j), h_0\right] = \exp\left[\sum_{j=1}^{m} t_jU_j, \log h_0\right]$.

For any $x \in G$, by the calculations above, there exists $h_0 \in G^{\alpha}$ and real numbers $s_l \in \mathbb{R}$ such that $x = h_0\prod\limits_{l=1}^{m}\exp(s_lU_l)$. For any $f \in L^2_{\mathbb{C}}(\mathbb{R}^m)$, $t, s \in \mathbb{R}^m$

$$(\pi'_{\alpha}(x)\,f)(t) = \left(\beta^{-1}f\right)\left(\prod_{j=1}^{m}\exp(t_jU_j)\,h_0\prod_{l=1}^{m}\exp(s_lU_l)\right).$$

Since $(\beta^{-1}f)(hg) = \overline{\alpha}(h)\,(\beta^{-1}f)(g)$, we see

$$(\pi'_{\alpha}(x)\,f)(t) = \overline{\alpha}\left(\left[\prod_{j=1}^{m}\exp(t_jU_j), h_0\right] h_0\exp\left(-\sum_{1\le j<l\le m} t_ls_j\,[U_j, U_l]\right)\right) f(t+s).$$

We have used the fact that $\exp([\mathfrak{g},\mathfrak{g}]) \subset Z(G) \subset G^{\alpha}$ and the calculations above. Since the restriction of B_{α} to $\mathbb{R}U_1 \oplus \ldots \oplus \mathbb{R}U_m \times \mathbb{R}U_1 \oplus \ldots \oplus \mathbb{R}U_m$ is zero, we have

$$(\pi'_{\alpha}(x)\,f)(t) = f(t+s)\,e^{2\pi i\alpha\left(\log h_0 + \left[\sum_{j=1}^{m} t_jU_j, \log h_0\right]\right)}.$$

Now, define the vector $w \in \mathbb{R}^m$ by

$$w := \left(\frac{\alpha(V_1)}{d_1}, ..., \frac{\alpha(V_m)}{d_m} \right).$$

Define the unitary isomorphism $T_w : L^2_{\mathbb{C}}(\mathbb{R}^m) \to L^2_{\mathbb{C}}(\mathbb{R}^m)$ by

$$(T_w f)(t) = f(t - w),$$

and define

$$\pi''_\alpha(x) = T_w \circ \pi'_\alpha(x) \circ T_w^{-1}$$

for all $x \in G$. We claim that the representation $\pi''_{\alpha *} = \rho_{\varepsilon *}$ is given by

$$\begin{aligned} \pi''_{\alpha *}(U_j) f(t) &= \frac{\partial}{\partial t_j} f(t), \\ \pi''_{\alpha *}(V_j) f(t) &= 2\pi i t_j d_j f(t), \\ \pi''_{\alpha *}(W_j) f(t) &= 2\pi i \alpha(W_j) f(t). \end{aligned} \tag{22}$$

To see this (see also [26, Section 3]), we have for $r \in \mathbb{R}$,

$$\begin{aligned} \pi''_\alpha(\exp(rU_j)f)(t) &= (T_w \pi'_\alpha(\exp(rU_j))T_{-w}f)(t) \\ &= (\pi'_\alpha(\exp(rU_j))T_{-w}f)(t-w) \\ &= (T_{-w}f)(t-w+re_j) \\ &= f(t+re_j), \end{aligned}$$

with e_j the j^{th} standard unit vector in $\mathbb{R}^m$. Also,

(1)

$$\begin{aligned} \pi''_\alpha(\exp(rV_j)f)(t) &= (\pi'_\alpha(\exp(rV_j))T_{-w}f)(t-w) \\ &= (T_{-w}f)(t-w)e^{2\pi i \alpha(rV_j + [(t_j - w_j)U_j, V_j])} \\ &= f(t)e^{2\pi i (r\alpha(V_j) + t_j d_j r - w_j d_j r)} \\ &= f(t)e^{2\pi i t_j d_j r}. \end{aligned}$$

We have

$$\begin{aligned} \pi''_\alpha(\exp(rW_j)f)(t) &= (\pi'_\alpha(\exp(rW_j))T_{-w}f)(t-w) \\ &= f(t)e^{2\pi i \alpha(rW_j)}. \end{aligned}$$

With $W_1 = Z_1, ..., W_{k_0} = Z_{k_0}$, equation (16) becomes (see (20))

$$\begin{aligned} D|_{\mathcal{H}_\alpha} &= \sum_{j=1}^{n} E_j \diamond \rho_{\varepsilon *}(E_j) + \frac{1}{4} \sum_{a \le k_0;\ b<i\le m_0} \langle Z_a, [X_b, X_i] \rangle (Z_a \diamond X_b \diamond X_i \diamond) \\ &= \sum_{j=1}^{n} E_j \diamond \rho_{\varepsilon *}(E_j) + M \\ &= \sum_{j=1}^{k_\alpha} (W_j \diamond) \rho_{\varepsilon *}(W_j) + \sum_{j=1}^{m} (U_j \diamond) \rho_{\varepsilon *}(U_j) + \sum_{j=1}^{m} (V_j \diamond) \rho_{\varepsilon *}(V_j) + M \\ &= \sum_{j=1}^{k_\alpha} 2\pi i \alpha (W_j)(W_j \diamond) + \sum_{j=1}^{m} (U_j \diamond) \frac{\partial}{\partial t_j} + \sum_{j=1}^{m} 2\pi i t_j d_j (V_j \diamond) + M \\ &= \sum_{j=1}^{m} (U_j \diamond) \frac{\partial}{\partial t_j} + \sum_{j=1}^{m} 2\pi i d_j (V_j \diamond) t_j + M'_\alpha, \end{aligned}$$

where M'_α is defined as the constant Hermitian transformation

$$M'_\alpha = M + \sum_{j=1}^{k_\alpha} 2\pi i \alpha (W_j)(W_j \diamond), \tag{23}$$

with M as in (20).

We have

$$\begin{aligned} D|_{\mathcal{H}_\alpha} &= M'_\alpha + \sum_{j=1}^{m} \left((U_j \diamond) \frac{\partial}{\partial t_j} + 2\pi i d_j (V_j \diamond) t_j \right) \\ &= M'_\alpha + \sum_{j=1}^{m} (U_j \diamond) \left(\frac{\partial}{\partial t_j} - 2\pi i d_j (U_j \diamond)(V_j \diamond) t_j \right), \end{aligned} \tag{24}$$

so that

$$D|_{\mathcal{H}_\alpha} = M'_\alpha + \sum_{j=1}^{m} (U_j \diamond) \left(\frac{\partial}{\partial t_j} + C_j t_j \right), \tag{25}$$

where we define

$$C_j = -2\pi i d_j (U_j \diamond)(V_j \diamond), \tag{26}$$

a Hermitian symmetric linear transformation.

4.3. **Matrix choices.** We now make specific choices of the matrices $(U_j \diamond)$, $(V_j \diamond)$, where U_j, V_r, W_k are from the basis chosen at the beginning of Section 4.2 relative to a particular α. We continue to use the positive real numbers d_j as defined in that section as well. Note that any other choices would yield the same Dirac spectrum.

Let

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\ \mathbf{1}' &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

We view $\Sigma_n = \mathbb{C}^{2^{\lfloor n/2 \rfloor}} = \bigotimes_{\lfloor n/2 \rfloor \text{ times}} \mathbb{C}^2$. Observe that multiplication satisfies

$$\begin{array}{|c|c|c|c|} \hline \mathbf{1} & \mathbf{1}' & \sigma_1 & \sigma_2 \\ \hline \mathbf{1}' & \mathbf{1} & -i\sigma_2 & i\sigma_1 \\ \hline \sigma_1 & i\sigma_2 & \mathbf{1} & -i\mathbf{1}' \\ \hline \sigma_2 & -i\sigma_1 & i\mathbf{1}' & \mathbf{1} \\ \hline \end{array} \tag{27}$$

with multiplication on the left given by the column items.

Let

$$(U_1 \diamond) = i\sigma_1 \otimes \mathbf{1} \otimes ... \otimes \mathbf{1}, \ (V_1 \diamond) = i\sigma_2 \otimes \mathbf{1} \otimes ... \otimes \mathbf{1},$$

and in general, for $1 \leq j \leq m$,

$$\begin{aligned} (U_j \diamond) &= i\mathbf{1}' \otimes ...\otimes \mathbf{1}' \otimes \sigma_1 \otimes \mathbf{1} \otimes ... \otimes \mathbf{1}, \\ (V_j \diamond) &= i\mathbf{1}' \otimes ...\otimes \mathbf{1}' \otimes \sigma_2 \otimes \mathbf{1} \otimes ... \otimes \mathbf{1}, \end{aligned} \tag{28}$$

where each $(U_j \diamond)$ and each $(V_j \diamond)$ has $j-1$ leading factors of $\mathbf{1}'$ and a total of $n' = \left\lfloor \frac{n}{2} \right\rfloor = m + \left\lfloor \frac{k_\alpha}{2} \right\rfloor$ matrix factors of size 2×2 . Continuing, each $(W_k \diamond)$, $1 \leq k \leq k_\alpha$, is chosen to be

$$(W_k \diamond) = i\mathbf{1}' \otimes ...\otimes \mathbf{1}' \otimes \sigma \otimes \mathbf{1} \otimes ... \otimes \mathbf{1}, \tag{29}$$

with σ being σ_1 or σ_2 according to whether k is odd or even, such that there are at least m leading factors of $\mathbf{1}'$ in the above expression. If the dimension k_α is odd, then the last matrix is chosen to be

$$(W_{k_\alpha} \diamond) = i\mathbf{1}' \otimes ...\otimes \mathbf{1}'. \tag{30}$$

With these choices, observe that from (26),

$$C_j = 2\pi d_j \left(\mathbf{1} \otimes \mathbf{1} \otimes ...\otimes \mathbf{1} \otimes \mathbf{1}' \otimes \mathbf{1} \otimes ... \otimes \mathbf{1} \right),$$

with $\mathbf{1}'$ in the j^{th} slot. We let

$$v_\ell = e_{\ell_1} \otimes e_{\ell_2} \otimes ... \otimes e_{\ell_{n'}} \tag{31}$$

where each $e_{\ell_\bullet}$ is either $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, with $\ell = (\ell_1, ..., \ell_{n'}) \in \{1, -1\}^{n'}$, then $\{v_\ell\}$ forms a basis of Σ_n. Then

$$\begin{aligned} C_j v_\ell &= 2\pi d_j \ell_j v_\ell, \\ (U_j \diamond) v_\ell &= i\ell_1 \ell_2 ... \ell_{j-1} v_{\ell^j} = \pm i v_{\ell^j} \ , \end{aligned}$$

with $\ell^j = (\ell_1, ..., -\ell_j, .., \ell_{n'})$.

We see that C_j commutes with every $C_{j'}$, and $C_j^2 = 4\pi^2 d_j^2 \mathbf{Id}$. Note that $\left\{ (2\pi d_j \ell_j, v_\ell) : \ell \in \{1, -1\}^{n'} \right\}$ is the set of eigenvalues and simultaneous orthonormal eigenvectors of every C_j, $j = 1, ..., m$.

Let $\mathbf{p} = (p_1, ..., p_m) \in \mathbb{Z}^m$. We let $h_{\mathbf{p}}(t) = h_{p_1}(t_1) ... h_{p_m}(t_m)$ using the Hermite functions

$$\begin{aligned} h_p(t) &= e^{t^2/2} \left(\frac{d}{dt} \right)^p e^{-t^2} \text{ for } p \geq 0, \\ h_p(t) &= 0 \text{ for } p < 0, \end{aligned} \tag{32}$$

which satisfy

$$h_p'(t) = th_p(t) + h_{p+1}(t)$$
$$h_{p+2}(t) + 2th_{p+1}(t) + 2(p+1)h_p(t) = 0.$$

The first equality is just the chain rule. To see the second equality, note that by the product rule and the binomial theorem,

$$\left(\frac{d}{dt}\right)^{p+2} e^{-t^2} = \left(\frac{d}{dt}\right)^{p+1}\left(-2te^{-t^2}\right) = -2t\left(\frac{d}{dt}\right)^{p+1} e^{-t^2} - 2(p+1)\left(\frac{d}{dt}\right)^{p} e^{-t^2},$$

and the result follows. Combining the two

$$h_p'(t) = th_p(t) - 2th_p(t) - 2ph_{p-1}(t) = -th_p(t) - 2ph_{p-1}(t).$$

Note that $\{h_{\mathbf{p}}(t) : \mathbf{p} \in (\mathbb{Z}_{\geq 0})^m\}$ is a basis of $L^2(\mathbb{R}^m, \mathbb{C})$.

For $\mathbf{p} \in \mathbb{Z}^m$, let

$$u_{\mathbf{p},\ell}(t) = h_{p_1}\left(\sqrt{2\pi d_1}t_1\right) h_{p_2}\left(\sqrt{2\pi d_2}t_2\right) ... h_{p_m}\left(\sqrt{2\pi d_m}t_m\right) v_\ell, \tag{33}$$

with v_ℓ as in (31). Observe that $u_{\mathbf{p},\ell} = 0$ if any coordinate of $\mathbf{p}$ is negative. Then, using the formulas above for $h_p'(t)$,

$$\begin{aligned}
\frac{\partial}{\partial t_j} u_{\mathbf{p},\ell}(t) &= -2\pi d_j t_j u_{\mathbf{p},\ell}(t) - 2p_j\sqrt{2\pi d_j}u_{\mathbf{p}-e_j,\ell}(t) \\
&= 2\pi d_j t_j u_{\mathbf{p},\ell}(t) + \sqrt{2\pi d_j}u_{\mathbf{p}+e_j,\ell}(t)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial}{\partial t_j} + t_jC_j\right) u_{\mathbf{p},\ell}(t) &= (2\pi d_j(\ell_j - 1))\, t_j u_{\mathbf{p},\ell}(t) - 2p_j\sqrt{2\pi d_j}u_{\mathbf{p}-e_j,\ell}(t) \\
&= (2\pi d_j(\ell_j + 1))\, t_j u_{\mathbf{p},\ell}(t) + \sqrt{2\pi d_j}u_{\mathbf{p}+e_j,\ell}(t)
\end{aligned}$$

Recall that $\mathbf{p}$ has dimension m, and ℓ has dimension $n' = \left\lfloor \frac{n}{2} \right\rfloor \geq m$. Now, from (25) we have

$$\begin{aligned}
Du_{\mathbf{p},\ell}(t) &= \left(M_\alpha' + \sum_{j=1}^{m} (U_j\diamond)\left(\frac{\partial}{\partial t_j} + t_jC_j\right)\right) u_{\mathbf{p},\ell}(t) \\
&= -2\sum_{j\leq m, \ell_j = 1} p_j\sqrt{2\pi d_j}\,(U_j\diamond)\, u_{\mathbf{p}-e_j,\ell}(t) \\
&\quad + \sum_{j\leq m, \ell_j = -1} \sqrt{2\pi d_j}\,(U_j\diamond)\, u_{\mathbf{p}+e_j,\ell}(t) + M_\alpha' u_{\mathbf{p},\ell}(t) \\
&= -2\sum_{j\leq m, \ell_j = 1} ip_j\sqrt{2\pi d_j}\ell_1\ell_2...\ell_{j-1}u_{\mathbf{p}-e_j,\ell^j}(t) \\
&\quad + \sum_{j\leq m, \ell_j = -1} i\sqrt{2\pi d_j}\ell_1\ell_2...\ell_{j-1}u_{\mathbf{p}+e_j,\ell^j}(t) + M_\alpha' u_{\mathbf{p},\ell}(t).
\end{aligned} \tag{34}$$

Often the eigensections can be found as linear combinations of the $u_{\mathbf{p},\ell}(t)$.

We modify the basis so that it is more convenient. For fixed $\mathbf{p} = (p_1, ..., p_m)$ and $\ell = (\ell_1, ..., \ell_m, ..., \ell_{n'})$, let $\mathbf{E}_\ell$ be the m-tuple defined by

$$(\mathbf{E}_\ell)_a = \begin{cases} 0 & \text{if } \ell_a = 1 \\ -1 & \text{if } \ell_a = -1 \end{cases}.$$

Then

$$\overline{u}_{\mathbf{p},\ell}(t) = \left(\prod_{j \le m,\ \ell_j = -1} \sqrt{2p_j} \right) u_{\mathbf{p}+\mathbf{E}_\ell,\ell}(t). \tag{35}$$

Using the fact that $\mathbf{p}+\mathbf{E}_\ell + e_j = \mathbf{p}+\mathbf{E}_{\ell^j}$ if $\ell_j = -1$ and $\mathbf{p}+\mathbf{E}_\ell - e_j = \mathbf{p}+\mathbf{E}_{\ell^j}$ if $\ell_j = 1$, we compute

$$\begin{aligned} D\overline{u}_{\mathbf{p},\ell}(t) &= -\sum_{j \le m, \ell_j = 1} 2i\sqrt{\pi d_j p_j}\ell_1\ell_2 \ldots \ell_{j-1}\overline{u}_{\mathbf{p},\ell^j}(t) \\ &\quad + \sum_{j \le m, \ell_j = -1} 2i\sqrt{\pi d_j p_j}\ell_1\ell_2 \ldots \ell_{j-1}\overline{u}_{\mathbf{p},\ell^j}(t) + M'_\alpha \overline{u}_{\mathbf{p},\ell}(t), \end{aligned}$$

so that

$$D\overline{u}_{\mathbf{p},\ell}(t) = -2i\sum_{j \le m} \sqrt{\pi d_j p_j}\ell_1\ell_2 \ldots \ell_j \overline{u}_{\mathbf{p},\ell^j}(t) + M'_\alpha \overline{u}_{\mathbf{p},\ell}(t). \tag{36}$$

5. Heisenberg Examples

Heisenberg Lie algebras are the only two-step nilpotent Lie algebras with one-dimensional center. Let $n = 2m+1$; define the n-dimensional Heisenberg Lie algebra by $\mathfrak{g} = \operatorname{span}\{X_1, \ldots, X_m, Y_1, \ldots, Y_m, Z\}$ with $[X_j, Y_k] = \delta_{jk} Z$ and other basis brackets not defined by skew-symmetry equal to zero. The n-dimensional Heisenberg Lie group G is the simply connected Lie group with Lie algebra $\mathfrak{g}$. A Heisenberg manifold is a quotient of G by a cocompact discrete subgroup Γ, where the metric comes from a left-invariant metric on G. From [18, Proposition 2.16], we see that every Heisenberg manifold is isometric to one with the following metric and lattice. The metric may be chosen for $\Gamma \backslash G$ on $(X_1, \ldots, X_m, Y_1, \ldots, Y_m, Z)$ to be

$$g_A = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \overline{g_A} & 0 \\ 0 & 1 \end{pmatrix}$$

where $A = \operatorname{diag}(a_1, \ldots, a_m)$ is a diagonal $m \times m$ matrix with positive nondecreasing entries.

We identify X_i with the matrix $E_{1,i+1}$, which is the matrix with 1 in the $(1, i+1)$-entry and all other entries zero. Similarly, we identify Y_j with $E_{j+1,m+2}$ and Z with $E_{m+2,m+2}$. In this section, we define $\exp(X_i)$ to be the matrix exponential $\exp(E_{1,i+1}) = I + E_{1,i+1}$, and we define $\exp(Y_j)$ and $\exp(Z)$ in a similar way. For $v \in \mathbb{R}^{2m}$ and $z \in \mathbb{R}$, we denote

$$(v, z) = \begin{pmatrix} 1 & v_1 & \ldots & v_m & z \\ 0 & 1 & \ldots & 0 & v_{m+1} \\ \vdots & \vdots & I & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & v_{2m} \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix},$$

With this notation,

$$\exp\left(x_1X_1+...+x_mX_m+y_1Y_1+...+y_mY_m+zZ\right)=\left(x_1,...,x_m,y_1,...,y_m,z+\frac{1}{2}x\cdot y\right),$$
$$\log\left(x_1,...,x_m,y_1,...,y_m,z\right)=x_1X_1+...+x_mX_m+y_1Y_1+...+y_mY_m+\left(z-\frac{1}{2}x\cdot y\right)Z. \tag{37}$$

To get from the matrix coordinates to the exponential coordinates, we use the change of coordinate mapping

$$(v,z)\mapsto \exp(v_1X_1+\ldots+v_mX_m+v_{m+1}Y_1+..+v_{2m}Y_m+(z-\frac{1}{2}(v_1v_{m+1}+\ldots+v_mv_{2m}))Z).$$

Every cocompact discrete subgroup Γ can be generated by $\exp(\mathcal{L})$ and $\exp(rZ)$, where $\mathcal{L}$ is a $2m$-dimensional lattice in $\mathbb{R}^{2m}=\operatorname{span}\{X_1,...,X_m,Y_1,...,Y_m\}$, and $\exp(rZ)$, $r>0$, generates a one-dimensional lattice in the center of G. We denote $\Gamma=\Gamma(\mathcal{L},r)$; note that not every choice of $(\mathcal{L},r)$ will yield a cocompact discrete subgroup. Two such Heisenberg manifolds determined by $(\mathcal{L},r,g_A)$ and $(\mathcal{L}',r',g_{A'})$ are isometric iff $g_A=g_{A'}$, $r=r'$, and there exists a matrix $\Phi\in\widetilde{Sp}(m,\mathbb{R})\cap O(2m,\overline{g_A})\subset M_{2m}(\mathbb{R})$ such that

$$\Phi(\mathcal{L})=\mathcal{L}'.$$

(See [18, Proposition 2.16]). Here, $O(2m,\overline{g_A})$ is the orthogonal group, and $\widetilde{Sp}(m,\mathbb{R})=\{\beta\in GL(2m,\mathbb{R}):\beta^tJ\beta=\pm J\}$, where $J=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}$.

5.1. **Three-dimensional case.**

5.1.1. *Eigenvalues.* For our Heisenberg manifold, we choose $\{X,Y,Z\}$ so that $[X,Y]=Z$ and $\left\{\frac{1}{\sqrt{A}}X,\frac{1}{\sqrt{A}}Y,Z\right\}$ is an orthonormal frame, with $A>0$. With notation as in the general case, we choose an element $\alpha\in\mathfrak{g}^*$, which fixes a coadjoint orbit.

Finite-dimensional irreducible subspaces: If the one-form $\alpha(Z)=0$, then $\mathfrak{g}_\alpha=\mathfrak{g}$, and $\mathfrak{g}^\alpha=\mathfrak{g}$ is the maximal polarizer of α. Then $G^\alpha=\exp(\mathfrak{g}^\alpha)=G$. Let

$$\mathcal{H}_\alpha=\{f:\mathfrak{g}\to\Sigma_n \mid \text{for some } s\in\Sigma_n, \text{all } h\in G, f(h)=\overline{\alpha}(h)\,s\ \},$$

where

$$\overline{\alpha}(h)=e^{2\pi i\alpha(\log h)}.$$

From (21),

$$\begin{aligned} D|_{\mathcal{H}_\alpha} &= \frac{2\pi i}{A}\alpha(X)(X\diamond)+\frac{2\pi i}{A}\alpha(Y)(Y\diamond)\\ &\quad+\frac{1}{4A^2}\langle Z,[X,Y]\rangle(Z\diamond X\diamond Y\diamond)\\ &= \frac{2\pi i}{A}\alpha(X)(X\diamond)+\frac{2\pi i}{A}\alpha(Y)(Y\diamond)+\frac{1}{4A^2}(Z\diamond X\diamond Y\diamond),\end{aligned}$$

which is a constant matrix. The eigenvalues of $D|_{\mathcal{H}_\alpha}$ are then the eigenvalues of this Hermitian matrix. We set

$$(X\diamond) = i\sqrt{A}\sigma_1 = \begin{pmatrix} 0 & i\sqrt{A} \\ i\sqrt{A} & 0 \end{pmatrix},\ (Y\diamond) = i\sqrt{A}\sigma_2 = \begin{pmatrix} 0 & -\sqrt{A} \\ \sqrt{A} & 0 \end{pmatrix},$$

$$(Z\diamond) = i\mathbf{1}' = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

The matrix is

$$D|_{\mathcal{H}_\alpha} = \begin{pmatrix} -\frac{1}{4A} & -\frac{2\pi}{\sqrt{A}}\left(\alpha\left(X\right)+i\alpha\left(Y\right)\right) \\ -\frac{2\pi}{\sqrt{A}}\left(\alpha\left(X\right)-i\alpha\left(Y\right)\right) & -\frac{1}{4A} \end{pmatrix}.$$

The eigenvalues are

$$\sigma_\alpha = \left\{-\frac{1}{4A}+2\pi\left\|\alpha\right\|, -\frac{1}{4A}-2\pi\left\|\alpha\right\|\right\}. \tag{38}$$

Infinite-dimensional irreducible subspaces: On the other hand, suppose $\alpha\left(Z\right) = \alpha\left(\left[X,Y\right]\right) = d$ is nonzero.

From (36), (23), and (31) we have, since $d_1 = \frac{|d|}{A}, m = 1, \ell = \pm 1, E_\ell = \frac{\ell-1}{2}$,

$$\begin{aligned} u_{p,\ell}\left(t\right) &= h_p\left(\sqrt{2\pi d_1}t\right)v_\ell, \\ \overline{u}_{p,\ell}\left(t\right) &= \begin{cases} u_{p,\ell}\left(t\right) & \ell=1 \\ \sqrt{2p}u_{p-1,\ell}\left(t\right) & \ell=-1 \end{cases}, \end{aligned}$$

$$D\overline{u}_{p,\ell}\left(t\right) = -2i\ell\sqrt{\pi\frac{|d|}{A}p}\,\overline{u}_{p,-\ell}\left(t\right) + M'_\alpha\overline{u}_{p,\ell}\left(t\right).$$

Then

$$\begin{aligned} M'_\alpha &= \frac{1}{4A^2}\left\langle Z,\left[X,Y\right]\right\rangle\left(Z\diamond X\diamond Y\diamond\right) + 2\pi i\alpha\left(Z\right)\left(Z\diamond\right) \\ &= \frac{1}{4A^2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & i\sqrt{A} \\ i\sqrt{A} & 0 \end{pmatrix}\begin{pmatrix} 0 & -\sqrt{A} \\ \sqrt{A} & 0 \end{pmatrix} + 2\pi id\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ &= \begin{pmatrix} -2\pi d-\frac{1}{4A} & 0 \\ 0 & 2\pi d-\frac{1}{4A} \end{pmatrix}, \end{aligned}$$

so that

$$M'_\alpha v_\ell = -\left(2\pi d\ell + \frac{1}{4A}\right)v_\ell,$$

and, for $\ell\in\{-1,1\}$, $p\in\mathbb{Z}_{\geq 0}$,

$$D\overline{u}_{p,\ell}\left(t\right) = -2i\ell\sqrt{\pi\frac{|d|}{A}p}\,\overline{u}_{p,-\ell}\left(t\right) + \left(-2\pi d\ell - \frac{1}{4A}\right)\overline{u}_{p,\ell}\left(t\right).$$

The $p=0$ case ($\overline{u}_{0,-1} = u_{-1,-1} = 0$) is

$$D\overline{u}_{0,1}\left(t\right) = \left(-2\pi d - \frac{1}{4A}\right)\overline{u}_{0,1}\left(t\right).$$

The matrix for D restricted to the span of $\{\overline{u}_{p,1},\overline{u}_{p,-1}\}$ for $p\geq 1$ is

$$\left(\begin{array}{cc} -2\pi d-\frac{1}{4A} & 2i\sqrt{\pi\frac{|d|}{A}p} \\ -2i\sqrt{\pi\frac{|d|}{A}p} & 2\pi d-\frac{1}{4A}\end{array}\right),$$

which has eigenvalues

$$-\frac{1}{4A}\pm 2\sqrt{\frac{\pi\left|d\right|p}{A}+\pi^{2}d^{2}}\,.$$

Thus, the list of all eigenvalues for the $\alpha\left(Z\right)=d\neq 0$ case is

$$\sigma_{\alpha}=\left\{-\frac{1}{4A}-2\pi d\right\}\cup\left\{-\frac{1}{4A}\pm 2\sqrt{\frac{\pi\left|d\right|p}{A}+\pi^{2}d^{2}}\ :p\geq 1\right\}.$$

5.1.2. *Occurrence conditions for lattice.* Here, the lattice $\mathcal{L}$ should be a two-dimensional lattice, say spanned by $v=(v_1,v_2)$ (corresponding to the matrix element $(v_1,v_2,0)$) and $w=(w_1,w_2)$. The central lattice is spanned by r (corresponding to $(0,0,r)$). Let $\widetilde{Sp}\left(1,\mathbb{R}\right)=\left\{\beta\in GL\left(2,\mathbb{R}\right):\beta^{t}J\beta=\pm J\right\}$. The condition $\beta^{t}J\beta=\pm J$ is equivalent to $\det\beta=\pm 1$, so in fact $\widetilde{Sp}\left(1,\mathbb{R}\right)\cap O\left(2,\mathbb{R}\right)=O\left(2,\mathbb{R}\right)$. This means we can rotate so that $v=(v_1,0)$ with $v_1>0$, and so that $w=(w_1,w_2)$ with $w_2>0$. Because v,w, and r generate a cocompact discrete subgroup, we must have, for any $h_1,h_2,h,k_1,k_2,k\in\mathbb{Z}$,

$$\begin{aligned}&\left(h_1v+h_2w,hr\right)\left(k_1v+k_2w,kr\right)\\ &\qquad=\left(v_1\left(h_1+k_1\right)+w_1\left(h_2+k_2\right),w_2\left(h_2+k_2\right),r\left(h+k\right)+h_1k_2v_1w_2+h_2k_2w_1w_2\right).\end{aligned}$$

is an element of the lattice, by closure for multiplication. Thus, for any choice of integers h_1,h_2,k_1,k_2, we must have $h_1k_2v_1w_2+h_2k_2w_1w_2\in r\mathbb{Z}$, i.e. $v_1w_2\ ,w_1w_2\in r\mathbb{Z}$. Letting $v_1w_2=rm_v$, $w_1w_2=rm_w$, we have $v=\left(\frac{rm_v}{w_2},0\right)$, $w=\left(\frac{rm_w}{w_2},w_2\right)$. The parameters are

$$A>0,r>0,w_2>0,m_v\in\mathbb{Z}_{>0},m_w\in\mathbb{Z}. \tag{39}$$

In our matrix coordinate system, from (37) we have

$$\begin{aligned}\log\left(\frac{rm_v}{w_2}h_1+\frac{rm_w}{w_2}h_2,w_2h_2,rh\right)&=\left(\frac{rm_v}{w_2}h_1+\frac{rm_w}{w_2}h_2\right)X+\left(w_2h_2\right)Y\\ &\qquad+\left(hr-\frac{1}{2}rh_1h_2m_v-\frac{1}{2}rh_2^2m_w\right)Z.\end{aligned}$$

The commutator satisfies

$$\left[\left(\frac{rm_v}{w_2}h_1+\frac{rm_w}{w_2}h_2,w_2h_2,rh\right),\left(\frac{rm_v}{w_2}k_1+\frac{rm_w}{w_2}k_2,w_2k_2,kr\right)\right]=\left(0,0,rm_v\left(h_1k_2-h_2k_1\right)\right).$$

We now determine a spin structure by fixing $\varepsilon:\Gamma\to\{1,-1\}$. Let

$$\begin{aligned}\varepsilon_1&=\varepsilon\left(v,0\right)=\varepsilon\left(\frac{rm_v}{w_2},0,0\right),\\ \varepsilon_2&=\varepsilon\left(w,0\right)=\varepsilon\left(\frac{rm_w}{w_2},w_2,0\right),\\ \varepsilon_3&=\varepsilon\left(0,0,r\right).\end{aligned}$$

Since $\varepsilon_1\varepsilon_2\varepsilon_1^{-1}\varepsilon_2^{-1} = (\varepsilon_3)^{m_v} = 1$ is the only relation, the values of ε_1 and ε_2 are arbitrary (± 1), but it may be that ε_3 is restricted by ${\varepsilon_3}^{m_v} = 1$. If m_v is even, there is no restriction, but

$$\text{if } m_v \text{ is odd, then } \varepsilon_3 = 1. \tag{40}$$

Now we choose an arbitrary element $\alpha \in \mathfrak{g}^*$, we may either choose $\alpha = \alpha_3 Z^*$ or $\alpha = \alpha_1 X^* + \alpha_2 Y^*$, since all possible coadjoint orbits may be parametrized by such elements. The occurrence condition is calculated on v and w. In particular, $\alpha(v)$ must be an integer or half integer depending on whether $\varepsilon_1 = \pm 1$. Likewise for $\alpha(w)$. From Section 7, the occurrence conditions are:

$$\begin{aligned} \alpha_1 \frac{r m_v}{w_2} &\in \mathbb{Z} + \frac{1-\varepsilon_1}{4}, & (41)\\ \alpha_1 \frac{r m_w}{w_2} + \alpha_2 w_2 &\in \mathbb{Z} + \frac{1-\varepsilon_2}{4}, & (42)\\ \alpha_3 r &\in \mathbb{Z} + \frac{1-\varepsilon_3}{4}. & (43) \end{aligned}$$

The multiplicities corresponding to these representations are as follows. If we choose $\alpha \in \mathfrak{g}^*$ such that $\alpha = \alpha_1 X^* + \alpha_2 Y^*$, then $m_\alpha = 1$ (see Section 7). If we choose $\alpha \in \mathfrak{g}^*$ such that $\alpha = \alpha_3 Z^*$, then $\mathfrak{g}_\alpha = \mathfrak{z}$. We have

$$m_\alpha = \sqrt{\det\left(B_\alpha|_{\operatorname{span}\{X,Y\}} \right)}$$

with respect to a lattice basis of $\mathcal{L}$, chosen to be $v = \frac{r m_v}{w_2} X$, $w = \frac{r m_w}{w_2} X + w_2 Y$, and thus

$$\begin{pmatrix} B_\alpha(v,v) & B_\alpha(v,w) \\ B_\alpha(w,v) & B_\alpha(w,w) \end{pmatrix} = \begin{pmatrix} 0 & \alpha([v,w]) \\ -\alpha([v,w]) & 0 \end{pmatrix}.$$

So

$$m_\alpha = |\alpha([v,w])| = |\alpha_3|\, r m_v \in \mathbb{Z}_{>0}.$$

The conditions (40) and (43) confirm that m_α is an integer.

Now we are ready to calculate the spectrum of the Dirac operator on a general Heisenberg 3-manifold with spin^c structure. Such a manifold with spin structure is given by $(\mathcal{L}, r, g_A, \varepsilon)$, and it is determined by the lattice basis $v = \frac{r m_v}{w_2} X$, $w = \frac{r m_w}{w_2} X + w_2 Y$ for $\mathcal{L}$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ as above with conditions (39), (40), (41), (42), (43).

We now calculate the part of the spectrum corresponding to each coadjoint orbit in $\mathfrak{g}^*$. There are two cases, $\alpha_3 = 0$ and $\alpha_3 \neq 0$. If $\varepsilon_3 = -1$, the condition (43) does not permit $\alpha_3 = 0$. As a consequence, finite dimensional irreducible subspaces do not occur. If $\varepsilon_3 = 1$, condition (43) is satisfied and $\alpha = \alpha_1 X^* + \alpha_2 Y^*$. The conditions (41) and (42) are satisfied if and only if there exist $j_1, j_2 \in \mathbb{Z}$ such that

$$\begin{aligned} \alpha_1 &= \frac{w_2}{r m_v}\left(j_1 + \frac{1-\varepsilon_1}{4} \right), \\ \alpha_2 &= \frac{1}{w_2}\left[\left(j_2 + \frac{1-\varepsilon_2}{4} \right) - \frac{m_w}{m_v}\left(j_1 + \frac{1-\varepsilon_1}{4} \right) \right], \end{aligned}$$

with eigenvalues

$$\begin{aligned}\sigma_\alpha &= \left\{-\frac{1}{4A}+2\pi\left\|\alpha\right\|,-\frac{1}{4A}-2\pi\left\|\alpha\right\|\right\}\\ &= \left\{-\frac{1}{4A}+2\pi\sqrt{\frac{\alpha_1^2+\alpha_2^2}{A}},-\frac{1}{4A}-2\pi\sqrt{\frac{\alpha_1^2+\alpha_2^2}{A}}\right\},\end{aligned}$$

and the multiplicity of this representation is $m_\alpha = 1$. If $\alpha = 0$ is permitted — i.e. $\varepsilon = \mathrm{id}$ — then $\mathcal{H}_\alpha$ is no longer irreducible, and the eigenspace corresponding to $-\frac{1}{4A}$ is two-dimensional.

We now consider the case $\alpha_3 \neq 0$. By (43) we may choose $\alpha \in \mathfrak{g}^*$ in the coadjoint orbit such that $\alpha = dZ^* = \frac{1}{r}\left(m+\frac{1-\varepsilon_3}{4}\right)Z^* \neq 0$ with $m \in \mathbb{Z}$, with eigenvalues

$$\left\{-\frac{1}{4A}-2\pi d\right\}\cup\left\{-\frac{1}{4A}\pm 2\sqrt{\frac{\pi\left|d\right|p}{A}+\pi^2d^2}:p\geq 1\right\},$$

or in other words

$$\begin{aligned}\sigma_\alpha = &\left\{-\frac{1}{4A}-\frac{2\pi}{r}\left(m+\frac{1-\varepsilon_3}{4}\right)\right\}\cup\\ &\left\{-\frac{1}{4A}\pm 2\sqrt{\frac{\pi p}{rA}\left|m+\frac{1-\varepsilon_3}{4}\right|+\frac{\pi^2}{r^2}\left(m+\frac{1-\varepsilon_3}{4}\right)^2}:p\in\mathbb{Z}_{>0}\right\},\end{aligned}$$

and the multiplicity of this representation is

$$m_\alpha = m_v\left|m+\frac{1-\varepsilon_3}{4}\right| > 0.$$

A special case occurs in [1], with $r = T'$, $A = (d')^2\,T'$, $m_v = r'$, $p = p'$, $d = \frac{\tau'}{T'}$, $m_w = 0$, where the primes indicate the notation used in [1].

5.2. **Eta invariant of three-dimensional Heisenberg manifolds.** From (11), the eta invariant of the spin^c Dirac operator corresponding to a spin structure on a three-dimensional manifold is ($n = 3, \widehat{n} = 2, W$ is trivial so that $\mathrm{tr}\left(F^W\right) = 0$)

$$\eta(0) = -\frac{\overline{\lambda}^3}{3\pi^2}\mathrm{vol}(M)+\frac{\overline{\lambda}}{24\pi^2}\int_M \mathrm{Scal} - 2\#\left(\sigma(D)\cap\left(\overline{\lambda},0\right)\right)-\#\left(\sigma(D)\cap\left\{0,\overline{\lambda}\right\}\right),$$

where $\overline{\lambda} < 0$ is the point of symmetry of the spectrum, and where the last two terms count multiplicities. (Recall the rank of the spinor bundle is two.)

We have

$$\overline{\lambda} = -\frac{1}{4A}$$

is the point of symmetry, and from ([14, Section 2]),

$$\mathrm{Scal} = \frac{1}{4}\mathrm{Tr}\left(j(Z)^2\right) = \frac{1}{4}\mathrm{Tr}\left(\begin{pmatrix} 0 & -\frac{1}{A}\\ \frac{1}{A} & 0\end{pmatrix}^2\right) = -\frac{1}{2A^2}.$$

Also,

$$\mathrm{vol}(M) = rA\det\begin{pmatrix}\frac{rm_v}{w_2} & \frac{rm_w}{w_2}\\ 0 & w_2\end{pmatrix} = r^2Am_v.$$

From the expressions for the eigenvalues of $\sigma(D)$, we see that $\#\left(\sigma(D)\cap\left\{-\frac{1}{4A}\right\}\right)$ is nonzero only if the part of the spectrum corresponding to $\alpha=0\in\mathfrak{g}^*$ is nontrivial. This happens only if $\varepsilon_1=\varepsilon_2=\varepsilon_3=1$. Thus,

$$\#\left(\sigma(D)\cap\left\{-\frac{1}{4A}\right\}\right)=\left\{\begin{array}{ll}2 & \text{if } \varepsilon=\mathbf{1}\\ 0 & \text{otherwise}\end{array}\right. .$$

To count $\#\left(\sigma(D)\cap\left(-\frac{1}{4A},0\right)\right)$, the eigenvalues coming from the toral eigenvalues are (see (38))

$$\begin{aligned}&\left\{-\frac{1}{4A}+\tau : 0<\tau=2\pi\sqrt{\frac{\alpha_1^2+\alpha_2^2}{A}}<\frac{1}{4A}\right\}\\ =\ &\left\{-\frac{1}{4A}+\tau : \tau=2\pi\frac{\|\alpha\|}{\sqrt{A}}, 0<\|\alpha\|<\frac{1}{8\pi\sqrt{A}}\right\}\end{aligned}$$

With fixed $r>0, w_2>0, m_v\in\mathbb{Z}_{>0}, m_w\in\mathbb{Z}$, by (41), (42) the coadjoint orbit represented by $\alpha=\alpha_1X^*+\alpha_2Y^*$ has an associated irreducible representation that occurs with multiplicity one if and only if

$$\begin{aligned}\frac{\alpha_1 r m_v}{w_2}&\in\mathbb{Z}+\frac{1-\varepsilon_1}{4},\\ \alpha_2w_2+\frac{rm_w}{w_2}\alpha_1&\in\mathbb{Z}+\frac{1-\varepsilon_2}{4}.\end{aligned}$$

The relevant nontoral eigenvalues are

$$\begin{aligned}\sigma_\alpha=&\left\{-\frac{1}{4A}-\frac{2\pi}{r}\left(m+\frac{1-\varepsilon_3}{4}\right) : m\in\mathbb{Z}, -\frac{r}{8\pi A}<m+\frac{1-\varepsilon_3}{4}<0\right\}\\ &\cup\left\{\begin{array}{c}-\frac{1}{4A}+2\sqrt{\frac{\pi p}{rA}\left|m+\frac{1-\varepsilon_3}{4}\right|+\frac{\pi^2}{r^2}\left(m+\frac{1-\varepsilon_3}{4}\right)^2} : p\in\mathbb{Z}_{>0}, m\in\mathbb{Z},\\ 0<rp\left|m+\frac{1-\varepsilon_3}{4}\right|+\pi A\left(m+\frac{1-\varepsilon_3}{4}\right)^2<\frac{r^2}{64\pi A}\end{array}\right\},\end{aligned}$$

with multiplicity $m_\alpha=m_v\left|m+\frac{1-\varepsilon_3}{4}\right|>0$. Letting $\mu=m+\frac{1-\varepsilon_3}{4}\in\mathbb{Z}+\frac{1-\varepsilon_3}{4}$, the inequality $0<rp|\mu|+\pi A\mu^2<\frac{r^2}{64\pi A}$ is equivalent to

$$0<|\mu|<\frac{r}{2\pi A}\left(\sqrt{\frac{1}{16}+p^2}-p\right),$$

so the relevant nontoral eigenvalues in the open interval $\left(-\frac{1}{4A},0\right)$ associated to $D|_{\mathcal{H}_\alpha}$ are

$$\begin{aligned}\sigma_\alpha=&\left\{-\frac{1}{4A}-\frac{2\pi}{r}\mu : \mu\in\mathbb{Z}+\frac{1-\varepsilon_3}{4}, -\frac{r}{8\pi A}<\mu<0\right\}\\ &\cup\left\{\begin{array}{c}-\frac{1}{4A}+2\sqrt{\frac{\pi p}{rA}|\mu|+\frac{\pi^2}{r^2}\mu^2} : p\in\mathbb{Z}_{>0}, \mu\in\mathbb{Z}+\frac{1-\varepsilon_3}{4},\\ 0<|\mu|<\frac{r}{2\pi A}\left(\sqrt{\frac{1}{16}+p^2}-p\right)\end{array}\right\},\end{aligned}$$

with multiplicities $m_\alpha=m_v|\mu|$.

In summary, summing over all coadjoint orbits whose associated irreducible representation occurs in ρ_ε ,

$$\begin{aligned}
\#\left(\sigma(D)\cap\left(-\frac{1}{4A},0\right)\right) &= \#\left\{\begin{array}{c}(\alpha_1,\alpha_2):\frac{\alpha_1 r m_v}{w_2}\in\mathbb{Z}+\frac{1-\varepsilon_1}{4},\\ \alpha_2 w_2+\frac{r m_w}{w_2}\alpha_1\in\mathbb{Z}+\frac{1-\varepsilon_2}{4},\ 0<\|\alpha\|<\frac{1}{8\pi\sqrt{A}}\end{array}\right\}\\
&\quad +m_v\sum_{\substack{\mu\in\mathbb{Z}+\frac{1-\varepsilon_3}{4}\\ -\frac{r}{8\pi A}<\mu<0}}|\mu|+m_v\sum_{p\in\mathbb{Z}_{>0}}\sum_{\substack{\mu\in\mathbb{Z}+\frac{1-\varepsilon_3}{4}\\ 0<|\mu|<\frac{r}{2\pi A}\left(\sqrt{\frac{1}{16}+p^2}-p\right)}}|\mu|.
\end{aligned}$$

Likewise,

$$\begin{aligned}
\#\left(\sigma(D)\cap\{0\}\right) &= \#\left\{\begin{array}{c}(\alpha_1,\alpha_2):\frac{\alpha_1 r m_v}{w_2}\in\mathbb{Z}+\frac{1-\varepsilon_1}{4},\\ \alpha_2 w_2+\frac{r m_w}{w_2}\alpha_1\in\mathbb{Z}+\frac{1-\varepsilon_2}{4},\ \|\alpha\|=\frac{1}{8\pi\sqrt{A}}\end{array}\right\}\\
&\quad +\left\{m_v|\mu| \text{ if } \mu=-\frac{r}{8\pi A}\in\mathbb{Z}+\frac{1-\varepsilon_3}{4}\right\}\\
&\quad +m_v\sum_{p\in\mathbb{Z}_{>0}}\sum_{\substack{\mu\in\mathbb{Z}+\frac{1-\varepsilon_3}{4}\\ |\mu|=\frac{r}{2\pi A}\left(\sqrt{\frac{1}{16}+p^2}-p\right)}}|\mu|.
\end{aligned}$$

We now show that the last line produces at most two nonzero terms. If $\mu_1,\mu_2\in\mathbb{Z}+\frac{1-\varepsilon_3}{4}$ both satisfy $|\mu_j|=\frac{r}{2\pi A}\left(\sqrt{\frac{1}{16}+p_j^2}-p_j\right)>0$ and $|\mu_1|\neq|\mu_2|$, solving for $\frac{r}{2\pi A}$ yields

$$k\left(\sqrt{1+16p_1^2}+4p_1\right)-h\left(\sqrt{1+16p_2^2}+4p_2^2\right)=0$$

for some positive $h,k\in\mathbb{Z}+\frac{1-\varepsilon_3}{4}$, and

$$1-\frac{h^2}{k^2}+32\frac{h}{k}p_1p_2-32\frac{h^2}{k^2}p_2^2=8\frac{h}{k}\left(\frac{h}{k}p_2-p_1\right)\sqrt{16p_2^2+1}.$$

If $\frac{h}{k}p_2=p_1$, then the equation above implies $p_1=p_2$. On the other hand,if $\left(\frac{h}{k}p_2-p_1\right)$ is not zero,

$$\frac{1-\frac{h^2}{k^2}+32\frac{h}{k}p_1p_2-32\frac{h^2}{k^2}p_2^2}{8\frac{h}{k}\left(\frac{h}{k}p_2-p_1\right)}=\sqrt{16p_2^2+1}.$$

Since the left side is rational and the right side is irrational, this is impossible.

Thus, there are at most two nonzero summands in the expression below.

$$\begin{aligned}
&m_v\sum_{p\in\mathbb{Z}_{>0}}\sum_{\substack{\mu\in\mathbb{Z}+\frac{1-\varepsilon_3}{4}\\ |\mu|=\frac{r}{2\pi A}\left(\sqrt{\frac{1}{16}+p^2}-p\right)}}|\mu|\\
&\quad=\begin{cases}\frac{m_v r}{2\pi A}\left(\sqrt{\frac{1}{16}+p^2}-p\right) & \text{if } \frac{r}{2\pi A}\left(\sqrt{\frac{1}{16}+p^2}-p\right)\ \in\mathbb{Z}+\frac{1-\varepsilon_3}{4} \text{ for some } p\in\mathbb{Z}_{>0}\\ 0 & \text{otherwise}\end{cases}
\end{aligned}$$

Then

$$\left\{ m_v \left|\mu\right| \text{ if } \mu = -\frac{r}{8\pi A} \in \mathbb{Z} + \frac{1-\varepsilon_3}{4} \right\} + m_v \sum_{p\in\mathbb{Z}_{>0}} \sum_{\substack{\mu\in\mathbb{Z}+\frac{1-\varepsilon_3}{4} \\ \mu=\pm\frac{r}{2\pi A}\left(\sqrt{\frac{1}{16}+p^2}-p\right)}} \left|\mu\right|$$
$$= \begin{cases} \frac{m_v r}{\pi A}\left(\sqrt{\frac{1}{16}+p^2}-p\right) & \text{if } \frac{r}{2\pi A}\left(\sqrt{\frac{1}{16}+p^2}-p\right) \in \mathbb{Z}+\frac{1-\varepsilon_3}{4} \text{ for some } p\in\mathbb{Z}_{>0} \\ \frac{m_v r}{8\pi A} & \text{if } \frac{r}{8\pi A} \in \mathbb{Z}+\frac{1-\varepsilon_3}{4} \\ 0 & \text{otherwise} \end{cases}$$

In summary,

$$\#\left(\sigma\left(D\right)\cap\{0\}\right) = \#\left\{ \begin{array}{c} \left(\alpha_1,\alpha_2\right) : \frac{\alpha_1 r m_v}{w_2} \in \mathbb{Z}+\frac{1-\varepsilon_1}{4}, \\ \alpha_2 w_2 + \frac{r m_w}{w_2}\alpha_1 \in \mathbb{Z}+\frac{1-\varepsilon_2}{4},\ \left\|\alpha\right\| = \frac{1}{8\pi\sqrt{A}} \end{array} \right\}$$
$$+ \begin{cases} \frac{m_v r}{\pi A}\left(\sqrt{\frac{1}{16}+p^2}-p\right) & \text{if } \frac{r}{2\pi A}\left(\sqrt{\frac{1}{16}+p^2}-p\right) \in \mathbb{Z}+\frac{1-\varepsilon_3}{4} \text{ for some } p\in\mathbb{Z}_{>0} \\ \frac{m_v r}{8\pi A} & \text{if } \frac{r}{8\pi A} \in \mathbb{Z}+\frac{1-\varepsilon_3}{4} \\ 0 & \text{otherwise} \end{cases}.$$

Putting these calculations together, we have

$$\begin{aligned} \eta\left(0\right) &= -\frac{\overline{\lambda}^3}{3\pi^2}\mathrm{vol}\left(M\right) + \frac{\overline{\lambda}}{24\pi^2}\int_M \mathrm{Scal} - 2\#\left(\sigma\left(-D\right)\cap\left(\overline{\lambda},0\right)\right) - \#\left(\sigma\left(-D\right)\cap\left\{0,\overline{\lambda}\right\}\right) \\ &= -\frac{\left(-\frac{1}{4A}\right)^3}{3\pi^2} r^2 A m_v + \frac{\left(-\frac{1}{4A}\right)}{24\pi^2}\left(-\frac{1}{2A^2}\right)\left(r^2 A m_v\right) - N\left(A,r,w_2,m_v,m_w,\varepsilon\right) \\ &= \frac{r^2 m_v}{192\pi^2 A^2} + \frac{r^2 m_v}{192\pi^2 A^2} - N\left(A,r,w_2,m_v,m_w,\varepsilon\right) \\ &= \frac{r^2 m_v}{96\pi^2 A^2} - N\left(A,r,w_2,m_v,m_w,\varepsilon\right), \end{aligned}$$

where $N(A,r,w_2,m_v,m_w,\varepsilon)$ is the nonnegative integer defined by

$$\begin{aligned}
N(\cdot) &= 2\#\left(\sigma(D)\cap\left(\overline{\lambda},0\right)\right)+\#\left(\sigma(D)\cap\left\{0,\overline{\lambda}\right\}\right)\\
&= 2\#\left\{\begin{array}{c}(\alpha_1,\alpha_2):\frac{\alpha_1 r m_v}{w_2}\in\mathbb{Z}+\frac{1-\varepsilon_1}{4},\\ \alpha_2 w_2+\frac{r m_w}{w_2}\alpha_1\in\mathbb{Z}+\frac{1-\varepsilon_2}{4},\ 0<\|\alpha\|<\frac{1}{8\pi\sqrt{A}}\end{array}\right\}\\
&\quad +2m_v\sum_{\substack{\mu\in\mathbb{Z}+\frac{1-\varepsilon_3}{4}\\ -\frac{r}{8\pi A}<\mu<0}}|\mu|+2m_v\sum_{p\in\mathbb{Z}_{>0}}\sum_{\substack{\mu\in\mathbb{Z}+\frac{1-\varepsilon_3}{4}\\ 0<|\mu|<\frac{r}{8\pi A\left(\sqrt{1+16p^2}+4p\right)}}}|\mu|\\
&\quad +\#\left\{\begin{array}{c}(\alpha_1,\alpha_2):\frac{\alpha_1 r m_v}{w_2}\in\mathbb{Z}+\frac{1-\varepsilon_1}{4},\\ \alpha_2 w_2+\frac{r m_w}{w_2}\alpha_1\in\mathbb{Z}+\frac{1-\varepsilon_2}{4},\ \|\alpha\|=\frac{1}{8\pi\sqrt{A}}\end{array}\right\}\\
&\quad +\begin{cases}\frac{m_v r}{\pi A}\left(\sqrt{\frac{1}{16}+p^2}-p\right) & \text{if } \frac{r}{2\pi A}\left(\sqrt{\frac{1}{16}+p^2}-p\right)\in\mathbb{Z}+\frac{1-\varepsilon_3}{4}\text{ for some } p\in\mathbb{Z}_{>0}\\ \frac{m_v r}{8\pi A} & \text{if } \frac{r}{8\pi A}\in\mathbb{Z}+\frac{1-\varepsilon_3}{4}\\ 0 & \text{otherwise}\end{cases}\\
&\quad +\begin{cases}2 & \text{if } \varepsilon_1=\varepsilon_2=\varepsilon_3=1\\ 0 & \text{otherwise}\end{cases}. \qquad (44)
\end{aligned}$$

All the sums above are finite.

We summarize this result in the following theorem.

Theorem 11. *The eta invariant of the spin^c Dirac operator on a three-dimensional Heisenberg manifold with parameters $A,r,w_2>0$, $m_v\in\mathbb{Z}_{>0}$, $m_w\in\mathbb{Z}$ with spin structure determined by $\varepsilon=(\varepsilon_1,\varepsilon_2,\varepsilon_3)\in\{\pm1\}^3$ satisfies*

$$\eta(0)=\frac{r^2 m_v}{96\pi^2A^2}-N(A,r,w_2,m_v,m_w,\varepsilon),$$

where $N(A,r,w_2,m_v,m_w,\varepsilon)$ is the nonnegative integer given by the expression (44).

Corollary 12. *From the expressions for $\eta(0)$, we may consider families of Heisenberg manifolds with constant $\eta(0)$. For example, if we let*

$$\begin{aligned} A &= b_1 r\\ w_2 &= b_2\sqrt{r}\end{aligned}$$

for some constants $b_1,b_2>0$. Holding m_v, m_w, ε, b_1, b_2 constant and letting r vary, we obtain a family of Heisenberg manifolds with constant $\eta(0)$ yet with different eigenvalues for D; even the point of symmetry $-\frac{1}{4A}$ varies with r.

Corollary 13. *Consider the "rectangular" Heisenberg 3-manifold (i.e. $m_w=0$). Suppose that the following conditions are met:*

(1) $A>\frac{r}{4\pi}$
(2) $\frac{r m_v}{4\pi\sqrt{A}}<w_2<4\pi\sqrt{A}$

Then if the spin structure is nontrivial ($\varepsilon\neq\mathrm{id}$),

$$\eta(0)=\frac{r^2 m_v}{96\pi^2A^2}.$$

Otherwise,

$$\eta(0) = \frac{r^2 m_v}{96\pi^2 A^2} - 2.$$

5.3. **Dirac Operator eigenvalues for general Heisenberg nilmanifolds .** We use the notation of Section 4. Suppose that $k_0 = 1$ is the dimension of the center $\mathfrak{z}$ and $n = 1 + m_0$ is the dimension of $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{v}$, and we will choose the orthonormal basis $\{Z, X_1, ..., X_{m_0}\}$ so that Z is a unit vector and $\{X_j\}$ is an orthonormal basis of $\mathfrak{v}$. From formula (16), the Dirac operator is

$$D = \sum_{i=1}^{n} (E_i \diamond) \rho_{\varepsilon *} (E_i) + \frac{1}{4} \sum_{b<i\leq m_0} \langle Z, [X_b, X_i] \rangle (Z \diamond X_b \diamond X_i \diamond),$$

acting on

$$\mathcal{H} = L^2 \left(\Gamma \diagdown G, G \times_\varepsilon \mathbb{C}^k \right) \cong L^2_\varepsilon (\Gamma \diagdown G) \otimes \Sigma_n,$$

which we decompose using Kirillov theory. Using notation from Section 4, the cases are:

Case 1: $k_\alpha = n$, i.e. $\alpha(Z) = 0$.

As in (21),

$$D|_{\mathcal{H}_\alpha} = \sum_{i=1}^{m_0} 2\pi i \alpha(X_i)(X_i \diamond) + \frac{1}{4} \sum_{b<i\leq m_0} \langle Z, [X_b, X_i] \rangle (Z \diamond X_b \diamond X_i \diamond), \tag{45}$$

which is a constant matrix. The eigenvalues of $D|_{\mathcal{H}_\alpha}$ are then the eigenvalues of this Hermitian matrix.

Case 2: $k_\alpha < n$, so that $\alpha(Z) \neq 0$.

For every noncentral vector v, there exists a vector w such that $B_\alpha(v, w) = \alpha([v, w]) = \alpha(Z) \neq 0$; we must have $\mathfrak{g}_\alpha = \mathfrak{z}$ and $k_\alpha = 1$. From (20), (23), (36), the Dirac operator may be expressed in terms of the basis $\{\overline{u}_{\mathbf{p},\ell}\}$ as

$$D\overline{u}_{\mathbf{p},\ell} = -\sum_j 2i\sqrt{\pi d_j p_j} \ell_1 \ell_2 ... \ell_j \overline{u}_{\mathbf{p},\ell^j} + M'_\alpha \overline{u}_{\mathbf{p},\ell},$$

where in this case

$$M'_\alpha \overline{u}_{\mathbf{p},\ell} = 2\pi i \alpha(Z)(Z\diamond)\overline{u}_{\mathbf{p},\ell} + \frac{1}{4}\sum_{j=1}^{m} \langle Z, [U_j, V_j] \rangle (Z \diamond U_j \diamond V_j \diamond) \overline{u}_{\mathbf{p},\ell}.$$

We use the matrix choices of Section 4.3, and for convenience, we let

$$Z = i\underbrace{\mathbf{1'} \otimes ... \otimes \mathbf{1'}}_{m \text{ times}},$$

and thus, since $\langle Z, [U_j, V_j] \rangle = d_j$, the formulas (28) and (27) yield

$$M'_\alpha \overline{u}_{\mathbf{p},\ell} = \left(-2\pi\alpha(Z)\ell_1...\ell_m - \frac{1}{4}\sum_{j\leq m} d_j \ell_1 ... \widehat{\ell_j} ... \ell_m \right) \overline{u}_{\mathbf{p},\ell}.$$

In summary,

$$D\overline{u}_{\mathbf{p},\ell} = -\sum_j 2i\sqrt{\pi d_j p_j}\ell_1\ell_2...\ell_j\overline{u}_{\mathbf{p},\ell^j} + \left(-2\pi\alpha(Z)\ell_1...\ell_m - \frac{1}{4}\sum_{j\leq m} d_j \ell_1 ... \widehat{\ell_j} ... \ell_m \right) \overline{u}_{\mathbf{p},\ell}, \tag{46}$$

and we have the following.

Proposition 14. *The infinite-dimensional subspace $\mathcal{H}_\alpha$ decomposes on any Heisenberg manifold as a direct sum of finite-dimensional subspaces that are invariant by the Dirac operator. In particular, the Dirac operator acts by the formula (46) on the finite-dimensional invariant subspace*

$$\mathcal{U}_{\mathbf{p}} = \operatorname{span}\left\{\overline{u}_{\mathbf{p},\ell} : \ell \in \{-1,1\}^m\right\}.$$

Remark 15. *For any specific example of a Heisenberg manifold, the formula (45) allows us to calculate the eigenvalues of D restricted to the finite-dimensional representations spaces $\mathcal{H}_\alpha$ with $\alpha(Z) = 0$, and the previous proposition allows us to calculate all other eigenvalues of D explicitly.*

5.4. **Vanishing of the eta invariant for $(2m+1)$-dimensional Heisenberg manifolds with m even.**

5.4.1. *Symmetry in the toroidal part of the spectrum for Heisenberg manifolds.* Suppose that we are given a $(2m+1)$-dimensional Heisenberg manifold, and $\alpha \in \mathfrak{g}^*$ is chosen so that $\alpha(Z) = 0$. Then we may choose an orthonormal basis $\{A_1, A_2, ..., A_m, B_1, B_2, ..., B_m\}$ of $\mathfrak{z}^\perp \subseteq \mathfrak{g}$ with the following properties:

(1) $\alpha(A_j) = 0$ if $j \geq 2$, $\alpha(B_j) = 0$ if $j \geq 1$;
(2) $[A_i, B_j] = a_j \delta_{ij} Z$ for some real numbers a_j.

(Simply choose A_1 orthogonal to $\ker\alpha$ and continue to form a symplectic basis of $\mathfrak{z}^\perp$.) Then the restriction of D to the subspace $\mathcal{H}_\alpha$ is

$$\begin{aligned} D|_{\mathcal{H}_\alpha} &= \sum_{i=1}^{m_0} 2\pi i\alpha(X_i)(X_i\diamond) + \frac{1}{4}\sum_{b<i\leq m_0} \langle Z, [X_b, X_i]\rangle (Z\diamond X_b \diamond X_i\diamond) \\ &= 2\pi i\alpha(A_1)(A_1\diamond) + \frac{1}{4}\sum_{j=1}^{m} a_j (Z\diamond A_j \diamond B_j\diamond). \end{aligned}$$

If m is even, then observe that $X_1 \diamond X_2 \diamond ...X_m\diamond$ anticommutes with $D|_{\mathcal{H}_\alpha}$ and is also invertible. Thus, it maps the λ eigenspace of $D|_{\mathcal{H}_\alpha}$ isomorphically onto the $-\lambda$ eigenspace of $D|_{\mathcal{H}_\alpha}$, and therefore the spectrum of $D|_{\mathcal{H}_\alpha}$ is symmetric about zero and does not contribute to the eta invariant.

A more complicated argument can be used to show that for all $m \geq 2$, the spectrum of $D|_{\mathcal{H}_\alpha}$ is symmetric about zero. Let

$$L_j = Z \diamond A_j \diamond B_j \diamond$$

for $1 \leq j \leq m$. Observe that L_j is symmetric, $L_j^2 = \mathbf{1}$, and $L_j L_k = L_k L_j$ for all j,k. Also $A_2\diamond$ is invertible and anticommutes with L_1, and $A_1\diamond$ anticommutes with L_j for $j > 1$. Thus, the dimension of the $+1$ eigenspace of L_j is the same as the dimension of the -1 eigenspace for L_j, and there exists a basis of simulaneous eigenvectors of $\Sigma_n = \mathbb{C}^{2^m}$. Let $\{v_1, ..., v_{2^{m-1}}\}$ be the subset of this basis consisting of $+1$ eigenvectors of L_2. Since A_2 commutes with L_2 and anticommutes with L_1, the $+1$-eigenspace of L_2 is a direct sum of $+1$ and -1 eigenspaces of L_1 in equal dimensions. Thus, we may further assume that $\{v_1, ..., v_{2^{m-2}}\}$ are $+1$-eigenvectors of L_1 and that $\{v_{2^{m-2}+1}, ..., v_{2^{m-1}}\}$ are -1-eigenvectors of L_1. Then $\{v_1, ..., v_{2^{m-1}}, iA_1v_1, ..., iA_1v_{2^{m-1}}\}$ provides a basis of $\mathbb{C}^{2^m}$ for which $D|_{\mathcal{H}_\alpha}$ corresponds to a

block matrix with 2^{m-2}-dimensional blocks of the form

$$x\begin{pmatrix} Q+R & 0 & I & 0 \\ 0 & -Q+R & 0 & I \\ I & 0 & Q-R & 0 \\ 0 & I & 0 & -Q-R \end{pmatrix},$$

where x is a scalar and Q and R are (commuting) diagonal matrices. A simple argument shows that the characteristic polynomial of such a matrix is an even function, and thus the spectrum of $D|_{\mathcal{H}_\alpha}$ is symmetric about zero and does not contribute to the eta invariant, if $m \geq 2$. We summarize the results in the following theorem.

Theorem 16. *On any Heisenberg manifold of dimension greater than* 3*, the restriction of the Dirac operator to any invariant subspace* $\mathcal{H}_\alpha$ *with* $\alpha(Z) = 0$ *has spectrum that is symmetric about* 0.

Remark 17. *No Heisenberg three-manifolds have this property; see (38).*

5.4.2. *Symmetry in the infinite-dimensional irreducible subspaces.* Next, suppose that $\alpha \in \mathfrak{g}^*$ is chosen so that $\alpha(Z) \neq 0$. Let $\mathcal{U} = \operatorname{span}\{\overline{u}_{\mathbf{p},\ell} : \mathbf{p} = (p_1, ..., p_m) \in (\mathbb{Z}_{\geq 0})^m, \ell \in \{-1,1\}^m\}$. Let $L : \mathcal{U} \to \mathcal{U}$ be the linear map defined by

$$L(\overline{u}_{\mathbf{p},\ell}) = \delta_\ell \overline{u}_{\mathbf{p},-\ell}\ ,$$

where $\delta_\ell = \pm 1$ according to an unspecified formula. Note that $L^{-1}(\overline{u}_{\mathbf{p},\ell}) = \delta_\ell \delta_{-\ell} L \overline{u}_{\mathbf{p},\ell} = \delta_{-\ell}\overline{u}_{\mathbf{p},-\ell}$. Now, we have

$$\begin{aligned}
L^{-1}DL\overline{u}_{\mathbf{p},\ell} &= \delta_\ell L^{-1} D \overline{u}_{\mathbf{p},-\ell} \\
&= \delta_\ell L^{-1}\begin{pmatrix} -\sum_j 2i\sqrt{\pi d_j p_j}\ell_1\ell_2...\ell_j(-1)^j\,\overline{u}_{\mathbf{p},-\ell^j} \\ +\left(-2\pi\alpha(Z)\ell_1...\ell_m(-1)^m - \frac{1}{4}\sum_{j\leq m} d_j \ell_1...\widehat{\ell_j}...\ell_m(-1)^{m-1}\right)\overline{u}_{\mathbf{p},-\ell}\end{pmatrix} \\
&= \delta_\ell\begin{pmatrix} -\sum_j 2i\sqrt{\pi d_j p_j}\ell_1\ell_2...\ell_j(-1)^j\,\delta_{\ell^j}\overline{u}_{\mathbf{p},\ell^j} \\ +\left(-2\pi\alpha(Z)\ell_1...\ell_m(-1)^m - \frac{1}{4}\sum_{j\leq m} d_j \ell_1...\widehat{\ell_j}...\ell_m(-1)^{m-1}\right)\delta_\ell\overline{u}_{\mathbf{p},\ell}\end{pmatrix} \\
&= \delta_\ell\left(-\sum_j 2i(-1)^j\delta_{\ell^j}\sqrt{\pi d_j p_j}\ell_1\ell_2...\ell_j\overline{u}_{\mathbf{p},\ell^j}\right) \\
&\quad + (-1)^{m-1}\left(2\pi\alpha(Z)\ell_1...\ell_m - \frac{1}{4}\sum_{j\leq m} d_j\ell_1...\widehat{\ell_j}...\ell_m\right)\overline{u}_{\mathbf{p},\ell}.
\end{aligned}$$

For the case where m is even, we define $\delta_\ell = (\ell_1)^2(\ell_2)^3...(\ell_m)^{m+1}$, so that $\delta_{\ell^j}\delta_\ell(-1)^j = (-1)^{j+1}(-1)^j = -1$, and thus, the matrix for $L^{-1}DL$ is the negative of the matrix for D with $\alpha(Z)$ replaced by its negative. Thus the spectrum σ_α satisfies $\sigma_{-\alpha} = -\sigma_\alpha$ if m is even. The following result follows due to the symmetry of the eigenvalues, since α occurs if and only if $-\alpha$ occurs; see (17).

Theorem 18. *Let* M *be a* $(4m'+1)$*-dimensional Heisenberg manifold with* $m' \in \mathbb{Z}_{>0}$*. Then the eta invariant of the spin*^{c} *Dirac operator associated to any spin*^{c} *structure is zero.*

Remark 19. *For the case where m is odd and the dimension is $2m+1$, we define $\delta_\ell = (\ell_1)^1 (\ell_2)^2 \ldots (\ell_m)^m$, so that $\delta_{\ell j}\delta_\ell (-1)^j = 1$; we see in that case that the matrix for $L^{-1}DL$ is the same as the matrix for D with $\alpha(Z)$ replaced by its negative. Thus the spectrum σ_α satisfies $\sigma_{-\alpha} = \sigma_\alpha$ if m is odd. Moreover, the eigenvalues of the Dirac operator need not be symmetric about 0, and the eta invariant need not be zero, as can be seen from the $m=1$ case in Section 5.2.*

6. Example of a five-dimensional non-Heisenberg nilmanifold

The purpose of this section is to exhibit an example of a two-step nilmanifold for which the techniques used above fail to produce the Dirac eigenvalues as eigenvalues of finite-dimensional matrices. We use the notation of Section 4 with a specific class of examples. We have that $k_0 = 2$ is the dimension of the center $\mathfrak{z}$ and $m_0 = 3$, and we have the orthonormal basis $\{Z_1, Z_2, X, Y_1, Y_2\}$ so that each Z_j is a unit vector and $\{X, Y_1, Y_2\}$ is an orthonormal basis of $\mathfrak{v}$. The only nontrivial bracket relations are $[X, Y_1] = Z_1$, $[X, Y_2] = Z_2$. From formula (16), the Dirac operator is

$$D = \sum_{i=1}^{5} \rho_{\varepsilon *}(E_i)(E_i \diamond) + \frac{1}{4} \sum_{i=1,2} Z_i \diamond X \diamond Y_i \diamond,$$

acting on

$$\mathcal{H} = L^2\left(\Gamma\diagdown G, G \times_\varepsilon \mathbb{C}^4\right) \cong L^2_\varepsilon(\Gamma\diagdown G) \otimes \Sigma_5,$$

which we decompose as follows. For $\alpha \in \mathfrak{g}^*$, the subspace $\mathcal{H}_\alpha$ of $L^2\left(\Gamma\diagdown G, G \times_\varepsilon \mathbb{C}^k\right)$ is invariant with respect to ρ_ε and invariant by D. If $\overline{\mathcal{H}_\alpha}$ is the irreducible ρ_ε-subspace of $L^2_\varepsilon(\Gamma\diagdown G)$ corresponding to the coadjoint orbit of α, we have $\mathcal{H}_\alpha \cong \overline{\mathcal{H}_\alpha} \otimes \Sigma_n$. As before, define the symplectic form on $\mathfrak{g}$ by $B_\alpha(u,v) = \alpha([u,v])$, and let $\mathfrak{g}_\alpha = \ker B_\alpha = \{u \in \mathfrak{g} : B_\alpha(u,\cdot) = 0\}$, $k_\alpha = \dim \mathfrak{g}_\alpha$.

6.1. Finite dimensional $\overline{\mathcal{H}_\alpha}$-irreducible subspaces: $k_\alpha = 5$, i.e. $\alpha(\mathfrak{z}) = 0$.

As in (21),

$$D|_{\mathcal{H}_\alpha} = 2\pi i\alpha(X)(X\diamond) + \sum_{j=1,2} 2\pi i \alpha(Y_j)(Y_j\diamond) + \frac{1}{4}\sum_{i=1,2} Z_i \diamond X \diamond Y_i \diamond .$$

The eigenvalues of $D|_{\mathcal{H}_\alpha}$ are then the eigenvalues of this constant Hermitian linear transformation.

We make the specific choices of the matrices $(E_j\diamond)$ as in Section 4.2. Note $\Sigma_n = \mathbb{C}^{2^2} = \mathbb{C}^2 \otimes \mathbb{C}^2$. We have

$$\begin{aligned} (X\diamond) &= i\mathbf{1}' \otimes \mathbf{1}',\ (Y_1\diamond) = i\sigma_1 \otimes \mathbf{1},\ (Y_2\diamond) = i\sigma_2 \otimes \mathbf{1}, \\ (Z_1\diamond) &= i\mathbf{1}' \otimes \sigma_1,\ (Z_2\diamond) = i\mathbf{1}' \otimes \sigma_2, \end{aligned}$$

Recalling (27), our matrix is (using basis $v_{1,1}$, $v_{-1,1}$, $v_{1,-1}$, $v_{-1,-1}$)

$$\begin{aligned} D|_{\mathcal{H}_\alpha} &= 2\pi i\alpha(X)(X\diamond) + \sum_{j=1,2} 2\pi i\alpha(Y_j)(Y_j\diamond) + \frac{1}{4}\sum_{i=1,2} Z_i \diamond X \diamond Y_i \diamond \\ &= -2\pi\alpha(X)\,\mathbf{1}'\otimes\mathbf{1}' - \sum_{j=1,2} 2\pi\alpha(Y_j)\,\sigma_j\otimes\mathbf{1} - \frac{i}{4}\sum_{j=1,2}(\mathbf{1}'\otimes\sigma_j)(\mathbf{1}'\otimes\mathbf{1}')(\sigma_j\otimes\mathbf{1}) \\ &= -2\pi\alpha(X)\,\mathbf{1}'\otimes\mathbf{1}' - \sum_{j=1,2} 2\pi\alpha(Y_j)\,\sigma_j\otimes\mathbf{1} + \frac{1}{4}(\sigma_1\otimes\sigma_2 - \sigma_2\otimes\sigma_1) \end{aligned}$$

$$= \begin{pmatrix} -2\pi\alpha(X) & -2\pi\alpha(Y_1) - i2\pi\alpha(Y_2) & 0 & 0 \\ -2\pi\alpha(Y_1) + i2\pi\alpha(Y_2) & 2\pi\alpha(X) & \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} & 2\pi\alpha(X) & -2\pi\alpha(Y_1) - i2\pi\alpha(Y_2) \\ 0 & 0 & -2\pi\alpha(Y_1) + i2\pi\alpha(Y_2) & -2\pi\alpha(X) \end{pmatrix}.$$

We may then determine that the four eigenvalues of $D|_{\mathcal{H}_\alpha}$ are:

$$\frac{1}{4} \pm \frac{1}{4}\sqrt{64\pi^2\alpha(X)^2 + 16\pi\alpha(X) + 64\pi^2\alpha(Y_1)^2 + 64\pi^2\alpha(Y_2)^2 + 1},$$
$$-\frac{1}{4} \pm \frac{1}{4}\sqrt{64\pi^2\alpha(X)^2 - 16\pi\alpha(X) + 64\pi^2\alpha(Y_1)^2 + 64\pi^2\alpha(Y_2)^2 + 1}.$$

Using the $\alpha \mapsto -\alpha$ symmetry, for a typical nilmanifold, this portion of the spectrum will be symmetric about zero.

6.2. **Infinite-dimensional $\overline{\mathcal{H}_\alpha}$-irreducible subspaces: $k_\alpha < n$, so that $\alpha(\mathfrak{z}) \neq 0$.** In this case, a typical coadjoint orbit has an element of the form $\alpha = b_2 Y_2^* + g_1 Z_1^* + g_2 Z_2^*$, with g_1, g_2 not both zero.

Choose a new orthonormal basis of $\mathfrak{g}$:

$$\left\{ W_1 = \frac{g_1 Z_1 + g_2 Z_2}{\sqrt{g_1^2 + g_2^2}}, W_2 = \frac{-g_2 Z_1 + g_1 Z_2}{\sqrt{g_1^2 + g_2^2}}, W_3 = \frac{-g_2 Y_1 + g_1 Y_2}{\sqrt{g_1^2 + g_2^2}}, U = X, V = \frac{g_1 Y_1 + g_2 Y_2}{\sqrt{g_1^2 + g_2^2}} \right\},$$

where $\{W_1, W_2, W_3\}$ is a basis of $\mathfrak{g}_\alpha$, $B_\alpha(U,U) = B_\alpha(V,V) = 0$, and

$$d := B_\alpha(U,V) = \alpha([U,V]) = \alpha\left(\left[X, \frac{g_1 Y_1 + g_2 Y_2}{\sqrt{g_1^2 + g_2^2}}\right]\right) = \sqrt{g_1^2 + g_2^2}.$$

Then the polarizing subalgebra $\mathfrak{g}^\alpha$ will be chosen to be

$$\mathfrak{g}^\alpha = \operatorname{span}\{V, W_1, W_2, W_3\},$$

and again $G^\alpha := \exp(\mathfrak{g}^\alpha)$.

Equation (24) becomes

$$D = (U\diamond)\frac{\partial}{\partial t} + 2\pi i d (V\diamond) t + M'_\alpha,$$

where M'_α is the constant Hermitian matrix (using $X_1 = U, X_2 = V, X_3 = W_3, W_1, W_2, k_0 = 2, m_0 = 3, k_\alpha = 3, m = 1$)

$$M'_\alpha = \frac{1}{4}\sum_{a\le 2;\ b<i\le 3} \langle W_a, [X_b, X_i]\rangle (W_a \diamond X_b \diamond X_i \diamond) + \sum_{j=1}^{3} 2\pi i \alpha(W_j)(W_j\diamond),$$

from (20), (23).

We calculate

$$[X_1, X_2] = [U,V] = W_1,\ [X_1, X_3] = W_2,\ [X_2, X_3] = 0,$$

so that

$$\begin{aligned} M &= \frac{1}{4}\sum_{a\le 2;\ b<i\le 3} \langle W_a, [X_b, X_i]\rangle (W_a \diamond X_b \diamond X_i \diamond) \\ &= \frac{1}{4} W_1 \diamond X_1 \diamond X_2 \diamond + \frac{1}{4} W_2 \diamond X_1 \diamond X_3 \diamond. \end{aligned}$$

Again we make the specific choices of the matrices $(E_j\diamond)$ as in Section 4.3, with

$$\begin{aligned}
(X_1\diamond) &= i\sigma_1\otimes\mathbf{1},\ (X_2\diamond)=i\sigma_2\otimes\mathbf{1},\ (X_3\diamond)=i\mathbf{1}'\otimes\mathbf{1}',\\
(W_1\diamond) &= i\mathbf{1}'\otimes\sigma_1,\ (W_2\diamond)=i\mathbf{1}'\otimes\sigma_2,
\end{aligned}$$

Then, since $\alpha(W_2)=0$, $\alpha(X_3)=\frac{g_1b_2}{d}$,$\alpha(W_1)=d=\sqrt{g_1^2+g_2^2}$, we use (27) to obtain

$$\begin{aligned}
M'_\alpha &= \frac{1}{4}W_1\diamond X_1\diamond X_2\diamond+\frac{1}{4}W_2\diamond X_1\diamond X_3\diamond+2\pi i\alpha(W_1)W_1\diamond\\
&\quad +2\pi i\alpha(X_3)(X_3\diamond)\\
&= \frac{1}{4}(i\mathbf{1}'\otimes\sigma_1)(i\sigma_1\otimes\mathbf{1})(i\sigma_2\otimes\mathbf{1})+\frac{1}{4}(i\mathbf{1}'\otimes\sigma_2)(i\sigma_1\otimes\mathbf{1})(i\mathbf{1}'\otimes\mathbf{1}')+2\pi id(i\mathbf{1}'\otimes\sigma_1)\\
&\quad +2\pi i\frac{g_1b_2}{d}(i\mathbf{1}'\otimes\mathbf{1}')\\
&= -\frac{1}{4}\mathbf{1}\otimes\sigma_1+\frac{1}{4}\sigma_1\otimes\sigma_1-2\pi d\mathbf{1}'\otimes\sigma_1-2\pi\frac{g_1b_2}{d}\mathbf{1}'\otimes\mathbf{1}'.
\end{aligned}$$

We need to determine what M'_α does to the basis $\{\overline{u}_{p,\ell}\}$. We have

$$\begin{aligned}
\overline{u}_{p,\ell} &= \begin{cases} u_{p,\ell} & \text{if } \ell_1=0\\ \sqrt{2p}\,u_{p-1,\ell} & \text{if } \ell_1=-1\end{cases},\\
u_{p,\ell} &= h_p\left(\sqrt{2\pi d}t\right)v_\ell
\end{aligned}$$

Then

$$\begin{aligned}
(\mathbf{1}\otimes\sigma_1)\overline{u}_{p,\ell} &= (\mathbf{1}\otimes\sigma_1)\begin{cases} u_{p,\ell} & \text{if } \ell_1=0\\ \sqrt{2p}\,u_{p-1,\ell} & \text{if } \ell_1=-1\end{cases}\\
&= \begin{cases} u_{p,\ell^2} & \text{if } \ell_1=0\\ \sqrt{2p}\,u_{p-1,\ell^2} & \text{if } \ell_1=-1\end{cases}\\
&= \overline{u}_{p,\ell^2}.
\end{aligned}$$

$$\begin{aligned}
(\sigma_1\otimes\sigma_1)\overline{u}_{p,\ell} &= (\sigma_1\otimes\sigma_1)\begin{cases} u_{p,\ell} & \text{if } \ell_1=0\\ \sqrt{2p}\,u_{p-1,\ell} & \text{if } \ell_1=-1\end{cases}\\
&= \begin{cases} u_{p,-\ell} & \text{if } \ell_1=0\\ \sqrt{2p}\,u_{p-1,-\ell} & \text{if } \ell_1=-1\end{cases}\\
&= \left(\sqrt{2p}\right)^{-\ell_1}\overline{u}_{p+\ell_1,-\ell}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{1}'\otimes\sigma_1)\overline{u}_{p,\ell} &= \ell_1(\mathbf{1}\otimes\sigma_1)\overline{u}_{p,\ell}\\
&= \ell_1\overline{u}_{p,\ell^2}
\end{aligned}$$

$$(\mathbf{1}'\otimes\mathbf{1}')\overline{u}_{p,\ell}=\ell_1\ell_2\overline{u}_{p,\ell}$$

Substituting,

$$
\begin{aligned}
M'_\alpha \overline{u}_{p,\ell} &= \left(-\frac{1}{4}\mathbf{1}\otimes\sigma_1 + \frac{1}{4}\sigma_1\otimes\sigma_1 - 2\pi d\mathbf{1}'\otimes\sigma_1 - 2\pi\frac{g_1 b_2}{d}\mathbf{1}'\otimes\mathbf{1}'\right)\overline{u}_{p,\ell} \\
&= -\frac{1}{4}\overline{u}_{p,\ell^2} + \frac{1}{4}\left(\sqrt{2p}\right)^{-\ell_1}\overline{u}_{p+\ell_1,-\ell} \\
&\quad -2\pi d\ell_1\overline{u}_{p,\ell^2} - 2\pi\frac{g_1 b_2}{d}\ell_1\ell_2\overline{u}_{p,\ell} \\
&= \left(-\frac{1}{4} - 2\pi d\ell_1\right)\overline{u}_{p,\ell^2} + \frac{1}{4}\left(\sqrt{2p}\right)^{-\ell_1}\overline{u}_{p+\ell_1,-\ell} - 2\pi\frac{g_1 b_2}{d}\ell_1\ell_2\overline{u}_{p,\ell}.
\end{aligned}
$$

From (36), we have

$$
\begin{aligned}
D\overline{u}_{p,\ell} &= -2i\sqrt{\pi dp}\,\ell_1\overline{u}_{p,\ell^1} + M'_\alpha\overline{u}_{p,\ell} \\
&= -2i\sqrt{\pi dp}\,\ell_1\overline{u}_{p,\ell^1} + \left(-\frac{1}{4} - 2\pi d\ell_1\right)\overline{u}_{p,\ell^2} \\
&\quad +\frac{1}{4}\left(\sqrt{2p}\right)^{-\ell_1}\overline{u}_{p+\ell_1,-\ell} - 2\pi\frac{g_1 b_2}{d}\ell_1\ell_2\overline{u}_{p,\ell}\ .
\end{aligned}
$$

There are no apparent invariant subspaces for D spanned by a finite number of the $\overline{u}_{p,\ell}$. The matrix for D is an infinite band matrix. This shows the difficulty of computing the Dirac eigenvalues for a general nilmanifold.

7. Appendix: CCMoore/LenRichardson Papers and Adaptations

7.0.1. *Occurrence and Multiplicity Condition.* Let Γ be a cocompact (i.e., $\Gamma\diagdown G$ compact) discrete subgroup of the simply connected nilpotent Lie group G. Let $\varepsilon : \Gamma \to \{\pm 1\} \subset GL\left(\mathbb{C}^k\right)$ be a homomorphism. Denote by U_ε the representation of G induced by ε; in particular,

$$U_\varepsilon = L^2_\varepsilon\left(\Gamma\diagdown G\right) = \left\{f : G \to \mathbb{C}^k : f\left(\gamma g\right) = \varepsilon\left(\gamma\right) f\left(g\right) \text{ for all } g \in G, \gamma \in \Gamma\right\},$$

where the (left) action of G on U_ε is given by interior right multiplication. Note that if $\varepsilon =$id, then $U_\varepsilon = L^2_\varepsilon\left(\Gamma\diagdown G\right)$ is the direct sum of k copies of the quasi-regular representation $U = L^2\left(\Gamma\diagdown G\right)$. As in the quasi-regular case, standard results in representation theory imply in general that U_ε can be decomposed into the direct sum of irreducible representations of G, each with finite multiplicity. A good reference for the standard representation theory used in this appendix is [12].

Our motivation for this construction is that spin^c structures over nilmanifolds $\Gamma\diagdown G$ correspond exactly to homomorphisms $\varepsilon : \Gamma \to GL\left(\mathbb{C}^k\right)$, where the image of ε lies in the set $\{\pm 1\}$, and $k = 2^{\lfloor n/2\rfloor}$. The resulting spinor bundle is $\Sigma_\varepsilon = G \times_\varepsilon \mathbb{C}^k = G \times \mathbb{C}^k \diagup \{(g,v) : (g,v) = (\gamma g, \varepsilon\left(\gamma\right) v) \text{ for all } \gamma \in \Gamma\}$; see [6, Prop 3.34, p. 114]. The sections of this bundle are elements of U_ε, on which the Dirac operator acts.

In the quasi-regular case ($\varepsilon = id$), L. Richardson and R. Howe, building on work of C. C. Moore, independently proved an exact occurrence condition and multiplicity formula; they determined the irreducible representations π of G that occur in $U = L^2\left(\Gamma\diagdown G\right)$ and the corresponding multiplicities $m\left(\pi, U\right)$. The purpose of this appendix is to generalize their occurrence and multiplicity formula from the quasi-regular to the case of general ε.

Before stating the main results, we require the following definitions and observations.

Denote by $\widehat{G}$ the set of equivalence classes of irreducible unitary representations of G. The Kirillov Correspondence is the bijection between the set of orbits of the co-adjoint action of G on $\mathfrak{g}^*$ and $\widehat{G}$. In particular, Kirillov Theory proves that to each $\alpha \in \mathfrak{g}^*$ corresponds an irreducible unitary representation π_α of G, every irreducible representation of G is unitarily equivalent to such a π_α, and two such irredicuble unitary representations π_α and $\pi_{\alpha'}$ are unitarily equivalent if and only if $\alpha' = \alpha \circ Ad(x^{-1})$ for some $x \in G$. Kirillov Theory applies mainly to nilpotent Lie groups, with generalizations to some solvable groups.

Choose $\alpha \in \mathfrak{g}^*$ and let $\mathfrak{h}$ be any subalgebra of $\mathfrak{g}$. Let $H = \exp(\mathfrak{h})$ be the unique simply connected Lie subgroup of G with Lie algebra $\mathfrak{h}$. The subalgebra $\mathfrak{h}$ or the subgroup H is *subordinate to* α iff $\alpha([\mathfrak{h},\mathfrak{h}]) \equiv 0$. If in addition $\mathfrak{h}$ is maximal with respect to being subordinate, then $\mathfrak{h}$ is called a *maximal subordinate subalgebra* for α, or a *polarizer* for α.

The explicit mapping between elements of $\mathfrak{g}^*$ and $\widehat{G}$ is as follows. Since G is nilpotent and simply connected, the exponential map is a diffeomorphism with inverse log. For $\alpha \in \mathfrak{g}^*$, let $\mathfrak{h}$ be a maximal subordinate subalgebra of α. Define $\overline{\alpha}(\cdot) = e^{2\pi i \alpha(\log(\cdot))}$, which is a character on H — i.e., a (complex) one-dimensional representation. The irreducible unitary representation π_α is the representation of G induced by the representation $\overline{\alpha}$ of H.

Recall that we have fixed a cocompact, discrete subgroup Γ of G. We call the pair $(\overline{\alpha}, H)$ *rational* (with respect to Γ) if it can be constructed with respect to a rational covector α, i.e. $\alpha(\log\Gamma) \subset \mathbb{Q}$.We call the pair $(\overline{\alpha}, H)$ a *special maximal pair* if $\log H = \mathfrak{h}$ is a maximal subordinate subalgebra for α that is special in the sense that it is algorithmically and inductively constructed from α and Γ as described in [28, pp. 176-178]. As Kirillov theory dictates that the representation π_α is independent of the maximal subordinate subalgebra (up to unitary equivalence), and as Richardson's paper shows that any covector α has a special maximal subordinate subalgebra, this additional property is not a restriction.We call $(\overline{\alpha}, H)$ an *ε-integral point* if and only if for all $\gamma \in \Gamma \cap H$, $\overline{\alpha}(\gamma) = e^{2\pi i \alpha(\log(\gamma))} = \varepsilon(\gamma)$. The equivalent condition on the Lie algebra level is, for all $\gamma \in \Gamma \cap H$,

$$\alpha(\log\gamma) \in \begin{cases} \mathbb{Z} & \text{if } \varepsilon(\gamma) = 1 \\ \mathbb{Z}+\frac{1}{2} & \text{if } \varepsilon(\gamma) = -1 \end{cases} .$$

Let $\pi \in \widehat{G}$ be induced from $\alpha \in \mathfrak{g}^*$ under the Kirillov correspondence. Let F be the family of special maximal characters of π, that is all possible pairs $(\overline{\alpha}, H)$ that induce π with $\mathfrak{h} = \log(H)$ a special maximal subordinate subalgebra. Now L. Richardson proved that $x \in G$ acts on F via

$$(\overline{\alpha}, H) \cdot x = \left(\overline{\alpha}^x, {}^{x^{-1}}H\right),$$

where I_x denotes conjugation by x, the function $\overline{\alpha}^x = \overline{\alpha} \circ I_x$, and ${}^{x^{-1}}H = x^{-1}Hx = I_{x^{-1}}(H)$. Note that $(\overline{\alpha}, H) \cdot x$ is an ε-integral point if and only if

$$\overline{\alpha}^x(\gamma) = \varepsilon(\gamma)$$

for all $\gamma \in \Gamma \cap \left({}^{x^{-1}}H\right)$ if and only if $\overline{\alpha}(\gamma) = \varepsilon(\gamma)$ for all $\gamma \in \left({}^{x^{-1}}\Gamma\right) \cap H$.

We may now state the following main results of this Appendix.

Theorem 20. *If π is induced by the special maximal character $(\overline{\alpha}, H)$ under the Kirillov correspondence, then $m(\pi, L^2_\varepsilon(\Gamma\diagdown G)) > 0$ if and only if there is an ε-integral point in the orbit $(\overline{\alpha}, H) \cdot G$.*

Lemma 21. *Assume that* $m\left(\pi, L^2_\varepsilon\left(\Gamma\diagdown G\right)\right) > 0$*, and let the special maximal character* $(\overline{\alpha}, H)$ *induce* π *under the Kirillov correspondence. The action satisifes* $(\overline{\alpha}, H)\cdot x = (\overline{\alpha}, H)$*, iff* $x \in H$*, so that we may identify the* G*-orbit of* $(\overline{\alpha}, H)$ *with* $H\diagdown G$*. If* $(\overline{\alpha}, H)$ *is an* ε*-integral point and if* $\gamma_0 \in \Gamma$*, then* $(\overline{\alpha}, H)\cdot\gamma_0$ *is also an* ε*-integral point.*

Let $(H\diagdown G)_\varepsilon$ be the set of ε-integral points in $H\diagdown G$. As a result of the Lemma, Γ acts on $(H\diagdown G)_\varepsilon$.

Theorem 22. *If the special maximal character* $(\overline{\alpha}, H)$ *induces* π *under the Kirillov correspondence, then the multiplicity of* π *in the* ε*-quasi regular representation* $U_\varepsilon = L^2_\varepsilon\left(\Gamma\diagdown G\right)$*, denoted* $m\left(\pi, L^2_\varepsilon\left(\Gamma\diagdown G\right)\right)$*, is the number of* Γ*-orbits in the set* $(H\diagdown G)_\varepsilon$ *of* ε*-integral points in the* G*-orbit* $H\diagdown G$ *of* $(\overline{\alpha}, H)$*. That is,*

$$m\left(\pi, L^2_\varepsilon\left(\Gamma\diagdown G\right)\right) = \#\left((H\diagdown G)_\varepsilon \diagup \Gamma\right).$$

7.0.2. *Proof of Occurrence and Multiplicity.* The proofs of the Lemma and Theorems follows the outline in L. Richardson's paper closely. We verify that a few key Lemmas of C. C. Moore extend to the ε-quasi-regular setting, and from there the proof primarily follows that of L.. Richardson verbatim, after substuting our Lemmas for those of Moore, and replacing "integral point" with "ε-integral point."

For any $\pi \in \widehat{G}$, suppose there exists $\gamma_0 \in \Gamma$ such that $N = \exp\left(\mathbb{R}\log\left(\gamma_0\right)\right)$ is a one-dimensional rational normal subgroup of G and $\pi(N) = 1$. Let φ be the natural projection of G onto $G_0 = G\diagup N$, so $\Gamma_0 = \Gamma\cdot N\diagup N = \varphi(\Gamma)$ is a cocompact discrete subgroup of G_0. Then the representation π_0 of G_0 defined by $\pi_0\left(\varphi(g)\right) = \pi(g)$ is well-defined and irreducible, hence an element of $\widehat{G_0}$ (see [25, Lemma 2.1]).

Lemma 23. *Generalized* ε*-Reduction Lemma. (generalization of* [25, Lemma 2.2]*, quoted as* [28, Lemma 2.6]*)*

Note that ε *induces a homomorphism of* Γ_0 *iff* $\varepsilon(\gamma_0) = 1$*. With notation as above, denote by* $U_{0\varepsilon}$ *the representation of* G_0 *induced by the* ε*-homomorphism of* Γ_0*, if it exists. If* $m(\pi, U_\varepsilon) \neq 0$ *then* $\varepsilon(\gamma_0) = 1$*, and the multiplicity* $m(\pi, U_\varepsilon)$ *of* π *in* U_ε *is equal to the multiplicity* $m(\pi_0, U_{0\varepsilon})$ *of* π_0 *in* $U_{0\varepsilon}$*.*

Proof. By normality, $N \subset Z(G)$. This follows from the Campbell-Baker-Hausdorff formula, since for vectors $A \in \log G$, and $X \in \log N$, we have $\exp(A)\exp(X)\exp(A^{-1}) = \exp(rX) = \exp\left(X + [A,X] + c_2[A,[A,X]] + ...\right)$. Let $\operatorname{ad}(A)^k(X)$ be the first zero element of the sequence $(X, [A,X], [A,[A,X]], ...)$. Because G is nilpotent, $\left\{X, [A,X], [A,[A,X]], ..., \operatorname{ad}(A)^{k-1}(X)\right\}$ is linearly independent. Since $rX = X + [A,X] + c_2[A,[A,X]] + ...$, we have $[A,X] = 0$. Note that since $\pi(N) = 1$, if $m(\pi, U_\varepsilon) \neq 0$, $U_\varepsilon(n) f = f$ for all n in N and f in the corresponding invariant subspace $\mathcal{H}_\pi$. This means that while $f(\gamma g) = \varepsilon(\gamma) f(g)$ for all $g \in G$, $\gamma \in \Gamma$, it must also be true that $(U_\varepsilon(n) f)(g) = f(gn) = f(ng) = f(g)$ for all $n \in N$. If $n \in \Gamma \cap N$, then in addition we have $f(gn) = f(ng) = f(g) = \varepsilon(n) f(g)$, which implies that $\varepsilon(\gamma)$ is the identity and ε induces a homomorphism of Γ_0. Thus ε restricted to $\Gamma \cap N$ acts trivially on the image of the sections of $\mathcal{H}_\pi$. Let $M = \Gamma\diagdown G$. We can project M onto $M_0 = \Gamma_0\diagdown G_0$, and M becomes a fiber bundle over M_0 with circle $T \cong (\Gamma \cap N)\diagdown N$ as fiber. Let

$$\mathcal{H}^N = \left\{f : G \to \mathbb{C}^k : f(\gamma g) = \varepsilon(\gamma) f(g) \text{ for all } \gamma \in \Gamma \text{ and } f(gn) = f(g) \text{ for all } n \in N\right\},$$

which is the set of sections on M that are constant on the fibers of $M \to M_0$, i.e. such that $U_\varepsilon(n) f = f$ for all $n \in N$. This is an invariant subspace of U_ε, because for such f,

$U_\varepsilon(n)\,U_\varepsilon(g)\,f = U_\varepsilon(g)\,U_\varepsilon(n)\,f = U_\varepsilon(g)\,f$ for all $g \in G$, $n \in N$. The projection of the space of all sections onto $\mathcal{H}^N$ lies in the center of the commuting algebra of U_ε; that is, the projection of U_ε onto an invariant subspace must commute everything that commutes with U_ε, because if L commutes with U_ε, then $\mathcal{H}^N$ is also an invariant subspace of L, and thus the projection onto $\mathcal{H}^N$ commutes with L. Let U_N be the restriction of U_ε to $\mathcal{H}^N$, and we define $U_{N_0}(\varphi(n)) = U_N(n)$, the corresponding representation of G_0. Using the realization of U_{N_0} on sections of M that are constant on the fibers, we can also realize U_{N_0} on the space $L^2(M_0, \Sigma_\varepsilon)$. It is clear that U_{N_0} is equivalent to $U_{0\varepsilon}$. We also have $m(\pi, U_\varepsilon) = m(\pi, U_N)$, since π is trivial on N, and $m(\pi, U_N) = m(\pi_0, U_{N_0}) = m(\pi_0, U_{0\varepsilon})$, as desired. $\square$

Lemma 24. *(Pukansky, as stated in* [28, Lemma 2.2]*) Let $\mathfrak{g}$ be a nilpotent Lie algebra with one dimensional center $\mathfrak{z} = \mathbb{R}Z_1$, with G and Γ as above. Then $\mathfrak{g} = \mathbb{R}X_1 \oplus \mathbb{R}Y_1 \oplus \mathbb{R}Z_1 \oplus \mathfrak{g}'$, where $[X_1, Y_1] = Z_1$. Let $\mathfrak{g}_1 = \mathbb{R}Y_1 \oplus \mathbb{R}Z_1 \oplus \mathfrak{g}' = \{X \in \mathfrak{g} : [X, Y_1] = 0\}$. The elements Y_1, Z_1 may be chosen to lie in $\log\Gamma$; i.e., $\mathfrak{g}_1$ may be chosen to be rational with respect to the cocompact discrete subgroup Γ of G.*

Theorem 25. *Kirillov's Theorem (as quoted in* [28, Theorem 2.3]*)*

If G has one dimensional center, then every irreducible representation π of G such that π is non-constant on the center is induced by a necessarily irreducible representation of $G_1 = \exp(\mathfrak{g}_1)$, with $\mathfrak{g}_1$ as in the previous Lemma.

Definition 26. *We call the subgroup G_1 from the previous theorem a* Kirillov subgroup.

Theorem 27. *ε-Generalized Moore's Algorithm (generalization of Moore's Algorithm, quoted as* [28, Moore's Algorithm 2.7]*).*
Let π be an irreducible representation of G, where G has a one dimensional center $Z(G)$, and $\pi|_{Z(G)} \neq \mathrm{id}$. Let π_1 be an irreducible representation of G_1, a rational Kirillov subgroup of G having codimension one, such that π_1 induces π. Define $\pi_1^x(x_1) = \pi_1(xx_1x^{-1})$ for x in G and x_1 in G_1. Let $U_{1\varepsilon}$ be defined for G_1 and $\Gamma_1 = \Gamma \cap G_1$ as U_ε is defined for G and Γ. Let $\widehat{G_1}$ denote the dual space of equivalence classes of unitary irreducible representations of G_1. Let $A' = \left\{\rho_1 \in \widehat{G_1} : m(\rho_1, U_{1\varepsilon}) > 0 \text{ and } \rho_1|_{Z(G)} \neq Id\right\}$. For all $\gamma \in \Gamma$, since G_1 is normal and $\gamma\Gamma\gamma^{-1} = \Gamma, U_{1\varepsilon}^\gamma \cong U_{1\varepsilon}$. Thus $m(\rho_1^\gamma, U_{1\varepsilon}) = m(\rho_1, U_{1\varepsilon})$, or $\{\rho_1^\gamma : \rho_1 \in A'\} = A'$.Let A be a subset of A' that meets each Γ-orbit in A' in exactly one element. Then

$$m(\pi, U_\varepsilon) = \sum_{\rho_1 \in \pi_1^G \cap A} m(\rho_1, U_{1\varepsilon}).$$

Proof. The proof closely follows that in [25, pp. 151–153].

Let $Z_2(G)$ be the subgroup of G such that $Z_2(G) \diagup Z(G)$ is the center of $G \diagup Z(G)$. The group $Z_2(G)$ is a rational subgroup of G (with respect to any lattice), see for example [12, Chapter 5], and we may choose a rational subgroup W of $Z_2(G)$ (and G) of dimension 2 that contains $Z(G)$. The centralizer G_0 of W then has codimension 1 in G and is a rational normal subgroup (see [28, Lemma 2.2], quoted from [22]). Finally, since G_0 is codimension one and normal, we can find a rational one-parameter group S such that $G = G_0 \rtimes S$.

We now use the following, whose proof can be found in any book on Kirillov theory. Denote by $U_{0\varepsilon}$the representation of G_0 induced by the homomorphism ε.

Lemma 28. *If $\pi \in \widehat{G}$ and if π is nontrivial on $Z(G)$, then π is induced by some $\pi_0 \in \widehat{G_0}$. The set of all representations of G_0 that induce π is the orbit of π_0 under G; that is, $\{\pi_0^x : x \in G\}$*

and $\pi_0^x = \pi_0^y$ *iff* $x = y \operatorname{mod} G_0$*, where* $\pi_0^x = \pi_0 \circ i_x$*. If* $\overline{\pi}$ *is the restriction of* π *to* G_0*, then* $\overline{\pi} = \int_{G/G_0} \pi_0^x \, dx = \int_S \pi_0^{x'} \, dx'$*, where* dx *and* dx' *refer to Haar measure in* $G/G_0 \cong S$*.*

Now let $(U_\varepsilon)^s$ be the subspace of U_ε complementary to the stabilizer of $U_\varepsilon|_{Z(G)}$. The projection onto the subspace corresponding to $(U_\varepsilon)^s$ is in the center of the commuting algebra of U_ε (see similar argument in the proof of Lemma 23). Thus if π is nontrivial on $Z(G)$ and occurs in U_ε, then it occurs in $(U_\varepsilon)^s$ just as often. Thus,

$$\begin{aligned} U_\varepsilon &= \sum_{\pi \in \widehat{G}} m(\pi, U_\varepsilon)\, \pi \\ (U_\varepsilon)^s &= \sum_{\pi \in B} m(\pi, U_\varepsilon)\, \pi, \end{aligned}$$

where B is the subset of $\widehat{G}$ consisting of those π that are nontrivial on $Z(G)$ and such that $m(\pi, U_\varepsilon) > 0$. For each $\pi \in B$, choose a $\pi_0 \in \widehat{G_0}$ that induces π. If $\overline{(U_\varepsilon)^s}$ is the restriction of $(U_\varepsilon)^s$ to G_0,

$$\begin{aligned} \overline{(U_\varepsilon)^s} &= \sum_{\pi \in B} m(\pi, U_\varepsilon)\, \overline{\pi} \\ &= \sum_{\pi \in B} m(\pi, U_\varepsilon) \int_{G/G_0} \pi_0^x \, dx. \end{aligned} \tag{47}$$

On the other hand, we can decompose $\overline{U_\varepsilon}$, the restriction of U_ε to G_0, by using Mackey's subgroup theorem. Indeed, let $U_{0\varepsilon}^x$ be the representation of G_0 induced by the ε-representation of $x\Gamma x^{-1} \cap G_0 = x(\Gamma \cap G_0)x^{-1}$ (since G_0 is normal). Note that as x is fixed, we can extend the definition of ε to $x\Gamma x^{-1}$. It is clear that $U_{0\varepsilon}^x$ is the conjugate by x of $U_{0\varepsilon}$; i.e., $U_{0\varepsilon}^x(n) = (U_{0\varepsilon})^x(n) = U_{0\varepsilon}(xnx^{-1})$. Then by Mackey's Theorem ([24, Theorem 12.1]), $U_{0\varepsilon}^x$ depends only on the double coset $\Gamma \cdot x \cdot G_0$ of x. But G_0 is normal, and $\Gamma \cdot x \cdot G_0 = \Gamma \cdot G_0 \cdot x$ is a coset of the subgroup ΓG_0. We know that ΓG_0 is closed (basic fact about nilpotent groups: Γ is cocompact discrete, G_0 is normal in G), and thus the double cosets fill out the group, allowing us to apply Mackey's Theorem.

Finally, (also by Mackey)

$$\overline{U_\varepsilon} = \int_{\Gamma \cdot G_0 \backslash G} U_{0\varepsilon}^y \, dy.$$

Now, if $\left(\overline{U_\varepsilon}\right)^s$ is the part of $\overline{U_\varepsilon}$ that is orthogonal to the stabilizer of $Z(G)$, then $\left(\overline{U_\varepsilon}\right)^s = \overline{(U_\varepsilon)^s}$, since the center is in G_0. Finally, if $(U_{0\varepsilon})^s$ is the similar subrepresentation of $U_{0\varepsilon}$ on which $Z(G)$ acts nontrivially, then one immediately deduces from the above that

$$\left(\overline{U_\varepsilon}\right)^s = \overline{(U_\varepsilon)^s} = \int_{\Gamma \cdot G_0 \backslash G}^{y} ((U_{0\varepsilon})^s)^y \, dy.$$

We write

$$(U_{0\varepsilon})^s = \sum_{\lambda_0 \in A'} m(\lambda_0, U_{0\varepsilon})\, \lambda_0,$$

where A' is the set of elements of $\widehat{G_0}$ that do not vanish on $Z(G)$ and for which $m(\lambda_0, U_{0\varepsilon}) > 0$. We are using the fact that $m(\lambda_0, U_{0\varepsilon}) = m(\lambda_0, (U_{0\varepsilon})^s)$ for $\lambda_0 \in A'$.

If $\gamma \in \Gamma$, then $\gamma\Gamma\gamma^{-1} \cap G_0 = \Gamma \cap G_0$, and from this it follows that $((U_{0\varepsilon})^s)^\gamma = (U_{0\varepsilon})^s$. Therefore, we have $m(\lambda_0^\gamma, U_{0\varepsilon}) = m(\lambda_0, U_{0\varepsilon})$, and thus $\gamma \cdot A' = A'$. Now let A be a subset of

A' such that A meets each orbit of Γ on A' in exactly one element. Since G_0 acts trivially on $\widehat{G_0}$ and hence on A', a ΓG_0-orbit in A' is just a Γ-orbit in A'. Moreover, G_0 (by Kirillov) is the subgroup of ΓG_0 leaving any point in A' fixed. Therefore, we can write

$$(U_{0\varepsilon})^s = \sum_{\lambda_0 \in A} m(\lambda_0, U_{0\varepsilon}) \sum_{s \in \Gamma \cdot G_0 \diagdown G_0} \lambda_0^s,$$

and thus

$$\left(\overline{U_\varepsilon}\right)^s = \sum_{\lambda_0 \in A} m(\lambda_0, U_{0\varepsilon}) \left[\int_{\Gamma \cdot G_0 \diagdown G} \left(\sum_{s \in \Gamma \cdot G_0 \diagdown G_0} \lambda_0^s \right)^y dy \right].$$

But since $G \diagup G_0$ is equivalent as a Borel space and measure space to $\Gamma \cdot G_0 \diagdown G \times (G_0 \diagdown \Gamma \cdot G_0)$ by choosing a Borel cross section, the representation in square brackets is just

$$\int_{G \diagup G_0} \lambda_0^x \, dx.$$

Thus,

$$\left(\overline{U_\varepsilon}\right)^s = \sum_{\lambda_0 \in A} m(\lambda_0, U_{0\varepsilon}) \int_{G \diagup G_0} \lambda_0^x \, dx. \tag{48}$$

Now, since G_0 is type I and direct integral decompositions are essentially unique, we may equate coefficients in (48) and (47). We find then that the family of orbits $\left\{\pi_0^G : \pi_0 \in B\right\}$ and $\left\{\lambda_0^G : \lambda_0 \in A\right\}$ are the same. Moreover, the orbits of π_0^G are all distinct, whereas some of the orbits of λ_0^G may coincide. Thus, we can equate the multiplicities as follows:

$$m(\pi, U_\varepsilon) = \sum_{\lambda_0 \in \pi_0^G \cap A} m(\lambda_0, U_{0\varepsilon}).$$

(End of Moore Algorithm Proof) □ □

Corollary 29. *Under the conditions of Moore's algorithm, $m(\pi, U_\varepsilon) > 0$ if and only if there is an irreducible representation π_1 of the rational Kirillov subgroup G_1 such that $m(\pi_1, U_{1\varepsilon}) > 0$ and $\pi = Ind_{G_1}^G(\pi_1)$.*

Remark 30. ***Abelian case:***
Suppose Γ is a lattice in $G = \mathbb{R}^n$, given by generators $\gamma_1, \gamma_2, ..., \gamma_n$. The coadjoint orbit of any $\alpha \in \mathfrak{g}^*$ is $\{\alpha\}$, and the maximal abelian subalgebra is $\mathfrak{h} = \mathfrak{g} = \mathbb{R}^n$. By the Kirillov correspondence this implies that irreducible representations of G are characters $x \mapsto e^{2\pi i \alpha(x)}$ of G determined by elements $\alpha \in \mathfrak{g}^* = (\mathbb{R}^n)^*$. Such an α occurs as a representation induced by ε if

$$e^{2\pi i \alpha(\gamma)} = \varepsilon(\gamma)$$

for all $\gamma \in \Gamma$.

This condition occurs exactly when $\alpha(\gamma) \in \mathbb{Z}$ whenever $\varepsilon(\gamma) = 1$ and $\alpha(\gamma) \in \mathbb{Z} + \frac{1}{2}$ when $\varepsilon(\gamma) = -1$; i.e., the pair $(\overline{\alpha}, H)$ is an ε-integral point. This means that there exists k_j, $l_j \in \mathbb{Z}$ such that

$$\alpha = \sum_{j,\ \varepsilon(\gamma_j)=1} k_j \gamma_j^* + \sum_{j,\ \varepsilon(\gamma_j)=-1} \left(\frac{1}{2} + l_j\right) \gamma_j^*,$$

where $\{\gamma_1^*, \gamma_2^*, ..., \gamma_n^*\}$ *is the basis of* $\mathfrak{g}^*$ *dual to* $\{\gamma_1, \gamma_2, ..., \gamma_n\}$*. So* π_α *can be written as*

$$\pi_\alpha(t) = \prod_{j,\ \varepsilon(\gamma_j)=1} e^{2\pi i k_j t_j} \prod_{j,\ \varepsilon(\gamma_j)=-1} e^{\pi i (2l_j+1)t_j},$$

with $t = \sum t_j \gamma_j \in \mathbb{C}^n$*.*

We now prove Theorem 20, the ε-generalized Richardson occurrence condition.

Proof. **Forward Direction:**
Suppose H has codimension zero. This implies that $\alpha([\mathfrak{g}, \mathfrak{g}]) = 0$, by the definition of maximal subordinate subalgebra. By possibly repeated application of Lemma 23, we can factor out $[\mathfrak{g}, \mathfrak{g}]$, and the occurrence and multiplicity remain unchanged. This reduces the problem to the abelian case, which is proved in Remark 30.

We now proceed inductively on the codimension of H: assume that the theorem is known for codimension $k-1$ or less. Now suppose $\pi \in \widehat{G}$ and that $m(\pi, U_\varepsilon)$ is greater than zero. Let π be induced from $(\overline{\alpha}, H)$, where the codimension of H is k.

Cases:

(1) Suppose that $\pi = 1$ on $Z(G)$. Since the center is always a rational subalgebra (for nilpotent groups, for any cocompact lattice), then we pick a one-dimensional rational subgroup $N \subset Z(G)$ on which π is trivial. Then we can apply Lemma 23, and we have reduced the codimension of H by one.

(2) Suppose that π acts nontrivially on $Z(G) \neq G$ and that $\dim(Z(G)) > 1$. We have that $U_\varepsilon(z)$ is multiplication by $\varepsilon(z)$ for all $z \in \Gamma \cap Z(G)$ by the definition of U_ε. Write $\pi = \pi_\lambda$ for some rational $\lambda \in \mathfrak{g}^*$. Since the kernel of λ restricted to $\mathfrak{z}$ is rational and at least dimension one, we can pick a one-dimensional rational subgroup $N \subset Z(G)$ on which π is trivial. We now apply Lemma 23 and reduce the codimension of H by one.

(3) Suppose that π acts nontrivially on $Z(G) \neq G$ and that $\dim(Z(G)) = 1$. Let G_1 be the rational Kirillov subgroup of G corresponding to π, and note that the codimension of G_1 is 1 and $H \subset G_1$, by construction. Let $U_{1\varepsilon}$ be the restriction of U_ε to G_1. By Corollary 29, there is an irreducible representation π_1 of G_1 such that $m(\pi_1, U_{1\varepsilon}) > 0$ and π_1 induces π. Let $\pi_1' = \mathrm{Ind}_H^{G_1}\overline{\alpha}$, which then induces π, and π_1' is also an irreducible representation of G_1 by the Kirillov theory. But π_1 must be equivalent to $\pi_1''(\cdot) = (\pi_1')^x(\cdot) := \pi_1(x(\cdot)x^{-1})$ for some $x \in G$ by the Kirillov correspondence. Since $m(\pi_1'', U_{1\varepsilon}) > 0$, there exists $g_1 \in G_1$ such that $f \circ \mathrm{Ad}(x) \circ \mathrm{Ad}(g_1) : \log(\Gamma \cap G_1) \to \mathbb{Q}$ (again, see [25, Cor. 2, p. 154]). Note that we do not know that $(\overline{\alpha}, H) \cdot x$ is maximal. Write $\log(xg_1) = aX_1 + P_1$, where $P_1 \in \mathfrak{g}_1$ and X_1 is the first external vector for $\mathfrak{h}$, as in the construction of the special maximal subordinate subalgebra in [28, Section 3]. Note that $Y_1 \in \log(\Gamma)$ from the construction satisfies $[X_1, Y_1] = Z_1 \in \log(\Gamma)$, which generates $Z(G)$. Since $\mathfrak{g}_1$ is the centralizer $C(Y_1, \mathfrak{g})$, we have

$$\begin{aligned} \alpha \circ \mathrm{Ad}(x) \circ \mathrm{Ad}(g_1)(Y_1) &= \alpha\left(Y_1 + [aX_1 + P_1, Y_1] + \frac{1}{2}[aX_1 + P_1, [aX_1 + P_1, Y_1]] ...\right) \\ &= \alpha(Y_1 + a[X_1, Y_1] + 0 + 0 + ...) \\ &= \alpha(Y_1 + aZ_1), \end{aligned}$$

by the Campbell-Baker-Hausdorff formula. Since $Y_1 \in \log(\Gamma)$, $\alpha(Y_1 + aZ_1) = \alpha(Y_1) + a\alpha(Z_1) \in \mathbb{Q}$, but since $Y_1, Z_1 \in \log(\Gamma)$ we have $a \in \mathbb{Q}$. Let $g_0 = \exp(aX_1)$. Then

$(\overline{\alpha}, H) \cdot g_0$ induces ρ_1 on G_1, which induces π on G, where $(\overline{\alpha}, H) \cdot g_0$ is a rational maximal character on G_1 and $m(\rho_1, U_{1\varepsilon}) > 0$. By construction, $(\overline{\alpha}, H) \cdot g_0$ is maximal.

By the induction hypothesis, there is an ε-integral point in $(\overline{\alpha}, H) \cdot g_0 \cdot G_1$, so that $(\overline{\alpha}, H) \cdot G$ has an ε-integral point.

Converse:

Suppose $(\overline{\alpha}, H) \cdot G$ has an ε-integral point $(\overline{\alpha}, H) \cdot g_0$. As above, we reduce to the case where the dimension of the center is 1 and π restricted to $Z(G)$ is nontrivial. We know that the Kirillov subgroup G_1 is normal in G, so our ε-integral point $(\overline{\alpha}, H) \cdot g_0$ induces $\pi_1^{g_0}$, which induces π^{g_0}, which is equivalent to π. Also, $(\overline{\alpha}, H)$ induces π_1, which induces π, and $(\overline{\alpha}, H) \cdot g_0$ is a maximal character in G_1. It follows from the induction hypothesis that $m(\pi_1^{g_0}, U_{1\varepsilon}) > 0$ which by Moore's induction implies that $m(\pi, U_\varepsilon) > 0$. □

Assume that $m(\pi, L^2_\varepsilon(\Gamma \diagdown G)) > 0$, and $(\overline{\alpha}, H)$ induces π, where $\log(H)$ is a special maximal subordinate subalgebra to α with respect to Γ. See [28, Section 3] for the construction for the special subordinate subalgebra.

Lemma 31. *If $x = \exp(X)$, and if $(\overline{\alpha}, H) \cdot x = (\overline{\alpha}, H)$, then $x \in H$.*

Proof. See [28, Section 5], Lemma 5.1. The proof holds verbatim. □

As a result, we may identify the G-orbit of $(\overline{\alpha}, H)$ with $H \diagdown G$.

Lemma 32. *If $(\overline{\alpha}, H)$ is an ε-integral point and if $\gamma_0 \in \Gamma$, then $(\overline{\alpha}, H) \cdot \gamma_0$ is an ε-integral point.*

Proof. Consider $(\overline{\alpha}, H) \cdot \gamma_0$. Note that $\Gamma \cap {}^{\gamma_0^{-1}} H = {}^{\gamma_0^{-1}} (\Gamma \cap H)$ since ${}^{\gamma_0^{-1}}\Gamma = \Gamma$. But if $\gamma_0^{-1}\gamma\gamma_0 \in \Gamma \cap {}^{\gamma_0^{-1}} H$, then $\overline{\alpha}^{\gamma_0}\left(\gamma_0^{-1}\gamma\gamma_0\right) = \overline{\alpha}(\gamma) = \varepsilon(\gamma) = \varepsilon\left(\gamma_0^{-1}\gamma\gamma_0\right)$ for every $\gamma \in \Gamma \cap H$. Also, ${}^{\gamma_0^{-1}}(\Gamma \cap H)$ is uniform in ${}^{\gamma_0^{-1}}H$. □

Let $(H \diagdown G)_\varepsilon$ be the set of ε-integral points in $H \diagdown G$. As a result of the second Lemma, Γ acts on $(H \diagdown G)_\varepsilon$.

We now prove Theorem 22.

Proof. The proof of [28, Theorem 5.3] goes through, replacing the reference to Lemma 2.6 with Lemma 23 and the reference to Lemma 2.7 with Theorem 27, and replacing the phrase "integral point" with "ε-integral point". □

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Department of Mathematics, University of Texas at Arlington, Arlington, Texas 76019-0408

E-mail address: rgornet@uta.edu

Department of Mathematics, Texas Christian University, Box 298900, Fort Worth, Texas 76129

E-mail address: k.richardson@tcu.edu